

THE DERIVED CATEGORY WITH RESPECT TO A GENERATOR

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ABSTRACT. Let \mathcal{G} be any Grothendieck category along with a choice of generator G , or equivalently a generating set $\{G_i\}$. We introduce the derived category $\mathcal{D}(G)$, which kills all G -acyclic complexes, by putting a suitable model structure on the category $\text{Ch}(\mathcal{G})$ of chain complexes. It follows that the category $\mathcal{D}(G)$ is always a well-generated triangulated category. It is compactly generated whenever the generating set $\{G_i\}$ has each G_i finitely presented, and in this case we show that two recollement situations hold. The first is when passing from the homotopy category $K(\mathcal{G})$ to $\mathcal{D}(G)$. The second is a G -derived analog of a recollement due to Krause. We describe several examples ranging from pure and clean derived categories to quasi-coherent sheaves on the projective line.

1. INTRODUCTION

This paper is about doing homological algebra with respect to a given generator in a Grothendieck category. Let R be a ring and $\text{Ch}(R)$ denote the category of chain complexes of (left) R -modules. Recall that the usual derived category $\mathcal{D}(R)$ is defined by first constructing the homotopy category $K(R)$ of unbounded chain complexes of R -modules, and then formally inverting the homology isomorphisms. R itself, when viewed as an R -module is a generator for $R\text{-Mod}$. But when R is viewed as a chain complex in degree zero, it is a weak generator for $\mathcal{D}(R)$ which essentially means it can detect exactness. Note that for a chain complex X , the standard isomorphism $\text{Hom}_R(R, X) \cong X$ allows one to view the homology of X as $H_n[\text{Hom}_R(R, X)]$. Similarly, homology isomorphisms can be viewed as those chain maps $X \rightarrow Y$ in $\text{Ch}(R)$ which become homology isomorphisms after applying $\text{Hom}_R(R, -)$.

But sometimes the derived category $\mathcal{D}(R)$ is not the right home for the homological algebra one is interested in. For example, there is the pure derived category of a ring R introduced in [CH02], and recently extended to any locally presented additive category in [Kra12]. Here if we take $G = \oplus G_i$ where the G_i range through a set of isomorphism representatives for all finitely presented objects, then a complex X is *pure acyclic* if and only if $H_n[\text{Hom}(G, X)]$ vanishes for all n . Similarly, isomorphisms in the pure derived category are those chain maps $X \rightarrow Y$ which become homology isomorphisms after applying $\text{Hom}(G, -) = \prod \text{Hom}(G_i, -)$. So we are essentially doing homological algebra with respect to the generator G .

The most important categories we encounter in homological algebra are the Grothendieck categories, which recall are the abelian categories having exact direct limits and a generator G . A generator G is equivalent to a generating set $\{G_i\}$ where $G = \oplus G_i$. This paper starts by showing that given any Grothendieck category \mathcal{G}

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and a fixed choice of generator $G = \oplus G_i$, we can define the derived category $\mathcal{D}(G)$. This category is obtained by inverting the *G-homology isomorphisms*, which are the chain maps $X \rightarrow Y$ in $\text{Ch}(\mathcal{G})$ such that $\text{Hom}_{\mathcal{G}}(G, X) \rightarrow \text{Hom}_{\mathcal{G}}(G, Y)$ is a homology isomorphism. Said another way, this is the category obtained from $\text{Ch}(\mathcal{G})$ by forcing the *G-acyclic complexes*, which are those complexes X for which $\text{Hom}_{\mathcal{G}}(G, X)$ is exact, to be 0. To do this, we begin by showing that the generator $G = \oplus G_i$ determines a Quillen exact structure on \mathcal{G} , which as we prove in Appendix B, is equivalent to a proper class of short exact sequences in the sense of [Mac63]. The short exact sequences here are precisely the usual short exact sequences which remain exact after applying $\text{Hom}_{\mathcal{G}}(G, -)$. We call them *G-exact sequences* and we denote this exact structure by \mathcal{G}_G . It becomes clear that we should define the *G-derived category* $\mathcal{D}(G)$ to be $\mathcal{D}(\mathcal{G}_G)$, the derived category with respect to the exact category \mathcal{G}_G , in the sense of [Nee90] and [Kel96].

But to get a deeper understanding of the *G-derived category* one would like to have a Quillen *model structure* on $\text{Ch}(\mathcal{G})$ whose trivial objects are the *G-acyclic complexes*. In this case the associated homotopy category would coincide with $\mathcal{D}(G)$. Such a model structure would first of all provide a convenient description of the morphism sets. But more importantly the theory of cofibrantly generated and monoidal model categories could be used to study $\mathcal{D}(G)$. We in fact are able to build not just one, but two cofibrantly generated models on $\text{Ch}(\mathcal{G})$ whose trivial objects are the *G-acyclic complexes*. The first is a generalization of the usual projective model structure on $\text{Ch}(R)$ while the second is a generalization of the usual injective model structure on $\text{Ch}(R)$. See [Hov99] for details on these model structures.

To summarize, we use Hovey's correspondence between cotorsion pairs and abelian model structures to obtain the following result.

Theorem A (Models for *G-derived categories*). *Let \mathcal{G} be any Grothendieck category with a generator $G = \oplus G_i$.*

- (1) *There is a model structure on $\text{Ch}(\mathcal{G})$ which we call the **G-projective model structure** whose trivial objects are the *G-acyclic complexes*. We call the associated homotopy category the **G-derived category**, and denote it by $\mathcal{D}(G)$. It is always a well generated triangulated category.*
- (2) *If each G_i is finitely presented then $\mathcal{D}(G)$ is compactly generated. In this case we also have a dual model structure on $\text{Ch}(\mathcal{G})$ which we call the **G-injective model structure**.*
- (3) *For given objects $A, B \in \mathcal{G}$ we have $\mathcal{D}(G)(A, \Sigma^n B) = \text{G-Ext}_{\mathcal{G}}^n(A, B)$ where $\text{G-Ext}_{\mathcal{G}}^n(A, B)$ denotes the group of (equivalence classes of) *n-fold G-exact sequences* $B \twoheadrightarrow X_1 \rightarrow \cdots \rightarrow X_n \twoheadrightarrow A$. *G-projective resolutions* (resp. *G-injective coresolutions*) provide cofibrant replacements in the *G-projective model structure* (resp. fibrant replacements in the *G-injective model structure*) and allow for computation of $\text{G-Ext}_{\mathcal{G}}^n(A, B)$ in the usual manner.*

It should be pointed out that a careful reading of [CH02] reveals that one can deduce the existence of the *G-projective model structure* above from their general Theorem 2.2. But while that theorem is more broad, our approach is different, and our results are very specific. To illuminate the analogy to the usual projective model structure on $\text{Ch}(R)$, where R is a ring, we give complete descriptions of the cofibrations, fibrations, and weak equivalences in the *G-projective model structure*. For example, the cofibrant objects are precisely the complexes P for which each P_n is a *G-projective* (direct summand of a coproduct of the G_i) and such that any chain

map $P \rightarrow X$, with target X G -acyclic, is null homotopic. The projective model structure is studied in Section 4. In particular, see Theorem 4.6 and Corollary 4.7 also Subsection 4.5.

On the other hand, constructing the dual G -injective model structure is far more technical than constructing the G -projective model. To do so we use the theory of purity from [AR94]. In particular, the assumption that the generating set $\{G_i\}$ satisfies that each G_i is finitely presented is equivalent to saying that \mathcal{G} is a locally finitely presented Grothendieck category. This is precisely the setting in which a nice theory of purity holds. See [AR94],[CB94], and Appendix A. We emphasize that this still includes the most important categories we encounter in homological algebra. For instance, the category of quasi-coherent sheaves over a quasi-compact and quasi-separated scheme is a locally finitely presented Grothendieck category by [Gar10, Proposition 3.1]. The injective model structure is studied in Section 5. In particular, see Subsection 5.4 with Theorem 5.11 being the dual of Theorem 4.6 and Corollary 5.12 being the dual of Corollary 4.7.

In Section 6 we go on to show that two recollement situations hold whenever we assume the G_i are finitely presented. Note that the G -projective and G -injective model structures are “balanced” in the sense that they share the same trivial objects. This is essentially the reason behind the following theorem. It is a G -derived version of a well known fact about $\mathcal{D}(R)$.

Theorem B (Verdier localization recollement for G -derived categories). *Suppose \mathcal{G} is a Grothendieck category and that $G = \oplus G_i$ is a generator with each G_i finitely presented. Let $\mathcal{D}(G)$ denote the G -derived category. Let $K(\mathcal{G})$ denote the homotopy category of all chain complexes and let $K_{G\text{-ac}}(\mathcal{G})$ denote the subcategory of all G -acyclic complexes. Then we have a recollement of triangulated categories:*

$$K_{G\text{-ac}}(\mathcal{G}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} K(\mathcal{G}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}(G) .$$

Proof. See Theorem 6.4 where the functors are described as well. \square

The existence of the injective model structure will also lead us to the following Theorem, which is a G -version of Krause’s result from [Kra05]. Here we call an object G -injective if it is injective with respect to the G -exact sequences already mentioned above.

Theorem C (Krause’s recollement for G -derived categories). *Let \mathcal{G} be a Grothendieck category and let $G = \oplus G_i$ be a generator with each G_i finitely presented. Let $\mathcal{D}(G)$ denote the G -derived category. Let $K_G(\text{Inj})$ denote the homotopy category of all complexes of G -injectives. Let $K_{G\text{-ac}}(\text{Inj})$ denote the homotopy category of all G -acyclic complexes of G -injectives. Then there is a recollement*

$$K_{G\text{-ac}}(\text{Inj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} K_G(\text{Inj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}(G) .$$

Proof. See Theorem 6.3. \square

The introduction continues in Section 2 where we list several applications or examples of the above Theorems.

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usual derived category with an alternate derived category obtained by “killing the G -acyclic complexes”. The author thanks him for the idea and for several helpful suggestions while writing the paper. The author also wishes to thank the referee for useful comments and suggestions.

2. EXAMPLES

As described in the Introduction, this paper shows that for a given set of generators $\{G_i\}$ in a Grothendieck category \mathcal{G} , we can do homological algebra by viewing everything “through the eyes of G ”. In particular, one should try to understand the proper class of G -exact sequences; those short exact sequences which remain exact after applying $\mathrm{Hom}_{\mathcal{G}}(G_i, -)$ for all the G_i . Whenever $G = \oplus G_i$ is projective, then this is just the usual class of short exact sequences and so $\mathcal{D}(G)$ is the usual derived category $\mathcal{D}(\mathcal{G})$. So the interesting thing is to explore what happens for other choices of G . We consider some examples here but there is much more room to explore this theme.

2.1. Pure and λ -pure derived categories. In [CH02], Christensen and Hovey put a model structure on $\mathrm{Ch}(R)$ whose homotopy category was the *pure derived category*, obtained by killing the pure acyclic complexes. More generally Krause shows in [Kra12, Theorem 4.1] that the pure derived category $\mathcal{D}_{\mathrm{pur}}(\mathcal{G})$ exists whenever \mathcal{G} is a locally finitely presentable Grothendieck category. In this case he shows there is a recollement situation when passing from $K(\mathcal{G})$ to $\mathcal{D}_{\mathrm{pur}}(\mathcal{G})$. This also follows from Theorem B by taking $G = \oplus G_i$ where the G_i range through a set of isomorphism representatives for all finitely presented objects. (However, we note that Krause does not even assume that \mathcal{G} is Grothendieck, merely additive.) But now we also have the following result as an immediate consequence of our above Theorem C.

Theorem D. *Suppose that \mathcal{G} is any locally finitely presentable Grothendieck category. Let $\mathcal{D}_{\mathrm{pur}}(\mathcal{G})$ denote the pure derived category. Let $K(\mathrm{PInj})$ denote the homotopy category of all complexes of pure-injective objects in \mathcal{G} . Let $K_{\mathrm{p-ac}}(\mathrm{PInj})$ denote the homotopy category of all pure acyclic complexes of pure-injectives. Then there is a recollement*

$$K_{\mathrm{p-ac}}(\mathrm{PInj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} K(\mathrm{PInj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}_{\mathrm{pur}}(\mathcal{G}) .$$

Theorem D is interesting, assuming our category \mathcal{G} admits chain complexes in $K_{\mathrm{p-ac}}(\mathrm{PInj})$ that are not contractible. We would like to have explicit examples of such complexes, or results indicating when such complexes do not exist.

We describe in Subsection 4.6 a generalization of the pure derived category to *any* Grothendieck category by replacing the notion of pure with the notion of λ -pure where λ is some large regular cardinal. We are only able to show that the projective model structure exists. But here a cofibrant replacement of an object $A \in \mathcal{G}$ is obtained by taking a λ -pure projective resolution of A in the sense of [Ros09]. It is worth noting that the existence of the λ -pure derived category doesn’t appear to follow from results in [Kra12] because the λ -pure short exact sequences are not closed under filtered colimits, only λ -filtered colimits. For a similar reason, the λ -pure exact structure on \mathcal{G} doesn’t appear to be, in general, of *Grothendieck type* in the sense of [Sto13]. We see in Subsection 4.6 that for any generator $G = \oplus G_i$, there is a regular cardinal λ and a canonical functor $\mathcal{D}_{\lambda\text{-pur}}(\mathcal{G}) \rightarrow \mathcal{D}(G)$ where

$\mathcal{D}_{\lambda\text{-pur}}(\mathcal{G})$ is the λ -pure derived category. This functor admits a left adjoint and provides a map of relative Ext groups $\lambda\text{-PExt}_{\mathcal{G}}^n(A, B) \rightarrow \text{G-Ext}_{\mathcal{G}}^n(A, B)$ which is natural in $A, B \in \mathcal{G}$.

2.2. Sheaves of modules on a ringed space. Let \mathcal{O}_X be a ringed space, that is, a sheaf of rings on a topological space X . The category $\mathcal{O}_X\text{-Mod}$ of sheaves of \mathcal{O}_X -modules is a Grothendieck category. Lets first recall the standard set of generators for $\mathcal{O}_X\text{-Mod}$. For each open $U \subseteq X$, extend \mathcal{O}_U by 0 outside of U to get a presheaf, which we denote by \mathcal{O}_U . Now sheafify to get an \mathcal{O}_X -module, which we will denote $j!(\mathcal{O}_U)$. There are standard isomorphisms $\text{Hom}(j!(\mathcal{O}_U), G) \cong \text{Hom}(\mathcal{O}_U, G) \cong G(U)$. It follows at once that the set $\{j!(\mathcal{O}_U)\}$ forms a generating set since the modules $j!(\mathcal{O}_U)$ “pick out points”. Hence the direct sum $G = \bigoplus_{U \subseteq X} j!(\mathcal{O}_U)$ is a generator. The above isomorphisms also imply that the G -exact category is just $\mathcal{O}_X\text{-Mod}$ together with the proper class of short *presheaf exact* sequences of \mathcal{O}_X -modules. That is, a G -exact sequence is an exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ of \mathcal{O}_X -modules for which $0 \rightarrow F(U) \rightarrow G(U) \rightarrow H(U) \rightarrow 0$ is an exact sequence of $\mathcal{O}(U)$ -modules for each open $U \subseteq X$. The G -derived category of Theorem A is thus the category of unbounded complexes of \mathcal{O}_X -modules modulo the the *presheaf acyclic* complexes. Using, again, the above isomorphisms, it follows immediately from [Har77, Exercise II.1.11] that each $j!(\mathcal{O}_U)$ is finitely presented whenever the space X is Noetherian. In particular, whenever $X = (X, \mathcal{O}_X)$ is a Noetherian scheme then $\mathcal{D}(G)$ is compactly generated. Also Theorems B and C apply in this case and the reader can interpret what they say. Just note that a G -injective \mathcal{O}_X -module here translates to one that is injective with respect to the short presheaf exact sequences. By Proposition 5.6, there are enough such G -injectives in the sense that we can find for any \mathcal{O}_X -module F a short presheaf exact sequence $0 \rightarrow F \rightarrow I \rightarrow I/F \rightarrow 0$ where I is G -injective.

2.3. Quasi-coherent sheaves over the projective line $\mathbf{P}^1(k)$. Let k be a commutative ring with identity. Here we consider the category of quasi-coherent sheaves over the projective line $\mathbf{P}^1(k)$. However, we use the quiver description of this category from [EE05], [EEGOB], [EEGOa] and [EEGR]. From this point of view, we consider the representation

$$R \equiv k[x] \hookrightarrow k[x, x^{-1}] \hookleftarrow k[x^{-1}]$$

of the quiver $Q \equiv \bullet \rightarrow \bullet \leftarrow \bullet$. Then R corresponds to the structure sheaf on $\mathbf{P}^1(k)$. A quasi-coherent sheaf of modules over $\mathbf{P}^1(k)$ may be thought of as a representation

$$A \equiv M \xrightarrow{f} L \xleftarrow{g} N$$

with M a $k[x]$ -module, L a $k[x, x^{-1}]$ -module, N a $k[x^{-1}]$ -module, f a $k[x]$ -linear map, and g a $k[x^{-1}]$ -linear map; all satisfying that the localization maps $S^{-1}f : S^{-1}M \rightarrow S^{-1}L \cong L$ and $T^{-1}g : T^{-1}N \rightarrow T^{-1}L \cong L$ are $k[x, x^{-1}]$ -isomorphisms, where $S = \{1, x, x^2, \dots\}$ and $T = \{1, x^{-1}, x^{-2}, \dots\}$. We call such an A a *quasi-coherent R -module*. A morphism is the obvious triple of linear maps providing commutative squares. Denote by $\text{Qco}(R)$ the category of all quasi-coherent R -modules. Then $\text{Qco}(R)$ is equivalent to the category of quasi-coherent sheaves on $\mathbf{P}^1(k)$ and so it is a Grothendieck category. There is a set of generators corresponding to the line bundles of degree n over $\mathbf{P}^1(k)$. They are the quasi-coherent R -modules

$$R(n) \equiv k[x] \hookrightarrow k[x, x^{-1}] \xleftarrow{x^n} k[x^{-1}], \quad n \in \mathbb{Z}$$

where the map on the right is multiplication by x^n . Tensor products, direct limits, and finite limits are all taken componentwise. In particular, a short exact sequence in $\text{Qco}(R)$ is one having all three involved short sequences exact. We refer the reader to [EE05], [EEGOB], [EEGOa] and [EEGR] for more detail on all of the above.

Now given any $A \in \text{Qco}(R)$, by regarding it as a diagram $M \xrightarrow{f} L \xleftarrow{g} N$ of just abelian groups, we may take the pullback $M \times_L N$. Denote this abelian group by PA . Also, given an integer n , denote by $A(n)$ the *twisted sheaf* $R(n) \otimes_R A$. Note that there is an obvious isomorphism $A(n) \equiv M \xrightarrow{f} L \xleftarrow{x^n \cdot g} N$. Each $R(n)$ is flat and in particular if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in $\text{Qco}(R)$, then so is $0 \rightarrow A(n) \rightarrow B(n) \rightarrow C(n) \rightarrow 0$. Consequently we have that $0 \rightarrow PA(n) \rightarrow PB(n) \rightarrow PC(n)$ is exact. If each $PB(n) \rightarrow PC(n)$ is also onto, then let's refer to $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ as a *twisted fibre exact sequence*.

From [EEGOa] we have that $\{R(n)\}$ is a set of (flat) generators for $\text{Qco}(R)$. Setting $G = \bigoplus_{n \in \mathbb{Z}} R(n)$, one can show that the G -exact sequences are precisely the twisted fibre exact sequences. Indeed for each n one can check directly that the elements of $\text{Hom}_{\text{Qco}(R)}(R(n), A)$ are in one to one correspondence with the elements of the pullback $PA(-n)$. That is, we have natural isomorphisms of abelian groups $\text{Hom}_{\text{Qco}(R)}(R(n), A) \cong PA(-n)$. This isomorphism also can be used to show that each $R(n)$ is finitely presented: For a direct limit $\varinjlim A_i$, using that pullbacks and tensor products commute with direct limits we see

$$\begin{aligned} \text{Hom}_{\text{Qco}(R)}(R(n), \varinjlim A_i) &\cong P[(\varinjlim A_i)(-n)] \cong P[R(-n) \otimes_R \varinjlim A_i] \cong \\ &P[\varinjlim (R(-n) \otimes_R A_i)] \cong \varinjlim P[R(-n) \otimes_R A_i] \cong \varinjlim PA_i(-n) \\ &\cong \varinjlim \text{Hom}_{\text{Qco}(R)}(R(n), A_i). \end{aligned}$$

So Theorems A, B, and C apply. Moreover, our characterization of the cofibrant and trivially cofibrant objects provided by Theorem 4.6 allows one to easily check that the model structure is *monoidal* so that the tensor product descends to a well-behaved tensor product on the G -derived category. To do this, apply Hovey's [Hov02, Theorem 7.2] and the method of [Gil07, Theorem 5.1]; it all boils down to the fact that $R(m) \otimes_R R(n) \cong R(m+n)$ which was shown from the quiver perspective in [EEGOa, Proposition 3.3].

2.4. Other examples concerning modules over a ring. Let R be a ring with 1, and let $\mathcal{G} = R\text{-Mod}$ be the category of (left) R -modules. Note that if \mathcal{S} is *any* set of R -modules, then $\mathcal{S} \cup \{R\}$ is a generating set for $R\text{-Mod}$. So Theorem A gives us a model structure killing the exact complexes which remain exact after applying $\text{Hom}_R(S, -)$ for all $S \in \mathcal{S}$. Of course Theorems B and C also hold if all the S are finitely presented modules. Moreover, whenever $\mathcal{S} \subseteq \mathcal{T}$, then in a way analogous to Corollary 4.8 we have a canonical functor $\mathcal{D}(\mathcal{T}) \rightarrow \mathcal{D}(\mathcal{S})$ with a left adjoint. The functor provides a mapping of relative Ext groups. We give two interesting examples below.

2.4.1. The clean derived category. For non-coherent rings we have the following variant of the pure derived category. An R -module is said to be of *type* FP_∞ if it has a projective resolution consisting of finitely generated free modules. The category of all type FP_∞ modules has a small skeleton. So we can take \mathcal{S} to be a set of isomorphism representatives. Then with $G = \bigoplus_{S \in \mathcal{S}} S$ we get that the G -exact category \mathcal{G}_G is exactly the category of R -modules along with the proper class

of all *clean exact sequences* in the sense of [BGH13]. The injectives in \mathcal{G}_G ought to be called *clean injective* modules. The projectives in \mathcal{G}_G are precisely direct summands of direct sums of modules of type FP_∞ . Since all modules of type FP_∞ are finitely presented, Theorems A, B and C apply giving recollements involving the *clean derived category*. We see a canonical functor from the pure derived category to the clean derived category. However, we point out that for coherent rings, a module is finitely presented if and only if it is of type FP_∞ . So this example only differs from the pure derived category for non-coherent rings.

It seems likely that the clean derived category will generalize to some other locally finitely presented Grothendieck categories. By [Bie81, Corollary 1.6] we have that for modules over a ring, F is of type FP_∞ if and only if $\text{Ext}_R^n(F, -)$ preserves direct limits for all $n \geq 0$. So in the more general setting, even without enough projective objects, one could define an object $F \in \mathcal{G}$ to be of type FP_∞ if $\text{Ext}_{\mathcal{G}}^n(F, -)$ preserves direct limits for all $n \geq 0$. However, one needs to be sure that the objects of type FP_∞ form a generating set for \mathcal{G} !

2.4.2. Inj-acyclic complexes. Suppose R is (left) Noetherian. Recall that every injective (left) R -module is a direct sum of indecomposable injective modules and there is a set \mathcal{S} of (isomorphism representatives) of all indecomposable injectives. (See [Lam99, Theorem 3.48].) So taking G to be the direct sum of R and all the indecomposable injectives, it is easy to see that a short exact sequence is G -exact if and only if it remains exact after applying $\text{Hom}_R(I, -)$ where I is any injective R -module. So these are a proper class of short exact sequences and the injective modules are projective objects with respect to these. More generally, by part (4) of Corollary 3.5, the G -projectives are precisely the direct summands of direct sums of modules in $\mathcal{S} \cup \{R\}$. By Theorem A, we get a model structure for an associated derived category obtained by killing all the exact “Inj-acyclic” complexes.

3. THE G -EXACT CATEGORY \mathcal{G}_G

Throughout this section \mathcal{G} will always denote a Grothendieck category with a chosen (fixed) set of generators $\{G_i\}_{i \in I}$. Furthermore, G will always denote their direct sum $G = \bigoplus_{i \in I} G_i$. So G itself is a generator for \mathcal{G} . The goal of this section is to give a detailed construction of an exact category, in the sense of Quillen [Qui73] and [Büh10], which we will call the G -exact category of \mathcal{G} . Being abelian, an exact structure on \mathcal{G} is, as shown in Appendix B, nothing more than a proper class of short exact sequences in the sense of [Mac63]. In this case, the proper class is the class of all G -exact sequences. That is, the short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ which remain exact after applying $\text{Hom}_{\mathcal{G}}(G, -)$. We denote this exact category by \mathcal{G}_G , and see that G is a projective generator for \mathcal{G}_G .

3.1. G -exact sequences and G -projectives. Recall that an object G in an abelian category \mathcal{A} is a *generator* if $\text{Hom}_{\mathcal{A}}(G, -)$ is faithful. Since \mathcal{A} is abelian this is equivalent to saying that if $f : A \rightarrow B$ is nonzero, then there exists a map $s : G \rightarrow A$ such that $fs \neq 0$. We have the following basic fact.

Lemma 3.1. *Let G be a generator for any abelian category \mathcal{A} and let X be a chain complex in $\text{Ch}(\mathcal{A})$. If the complex of abelian groups $\text{Hom}_{\mathcal{A}}(G, X)$ is exact, then X itself must be exact.*

Proof. We just need to show that $d_{n+1} : X_{n+1} \rightarrow Z_n X$ is an epimorphism, that is, right cancelable. Since \mathcal{A} is abelian we just need to show that for a map $f : Z_n X \rightarrow Y$ we have $f d_{n+1} = 0$ implies $f = 0$. By way of contradiction, say $f d_{n+1} = 0$ but $f \neq 0$. Then because G is a generator we get a map $s : G \rightarrow Z_n X$ such that $f s \neq 0$. But notice s determines a map in the domain of $(d_n)_* : \text{Hom}_{\mathcal{A}}(G, X_n) \rightarrow \text{Hom}_{\mathcal{A}}(G, X_{n-1})$ for which $(d_n)_*(s) = 0$. So by hypothesis we have $s \in \ker (d_n)_* = \text{Im } (d_{n+1})_*$ which ensures a map $t : G \rightarrow X_{n+1}$ such that $s = d_{n+1} t$. Now $f d_{n+1} = 0$ implies $f d_{n+1} t = 0$ implies $f s = 0$, which is the contradiction. \square

Now let $R = \text{Hom}_{\mathcal{G}}(G, G)$ be the endomorphism ring of G and let $\text{Mod-}R$ be the category of right R -modules. By the Gabriel-Popescu Theorem, the functor $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \rightarrow \text{Mod-}R$ is fully faithful and has an exact left adjoint T . Therefore \mathcal{G} is equivalent to the full subcategory $\mathcal{S} = \text{Im}[\text{Hom}_{\mathcal{G}}(G, -)]$ of $\text{Mod-}R$. Since the property of being a Grothendieck category is stable under equivalence of categories we know that \mathcal{S} is Grothendieck. However \mathcal{S} is not an abelian subcategory of $\text{Mod-}R$. In particular, if $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence in \mathcal{G} , then of course $0 \rightarrow \text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{G}}(G, C) \rightarrow 0$ is generally only a left exact sequence in $\text{Mod-}R$. But this IS a short exact sequence in the abelian category \mathcal{S} . Indeed let's show directly that g_* is right cancelable in \mathcal{S} , making it an epimorphism in \mathcal{S} : Given any morphism $t : \text{Hom}_{\mathcal{G}}(G, C) \rightarrow S$ in \mathcal{S} , we wish to show $0 = t g_*$ implies $0 = t$. But $\text{Hom}_{\mathcal{G}}(G, -)$ is full and so t must take the form $\text{Hom}_{\mathcal{G}}(G, C) \xrightarrow{h_*} \text{Hom}_{\mathcal{G}}(G, D)$ for some $h : C \rightarrow D$ in \mathcal{G} . So we have $0 = t g_* = h_* g_* = (h g)_*$. Since $\text{Hom}_{\mathcal{G}}(G, -)$ is faithful we have $h g = 0$. But g is right cancelable, so $h = 0$ and this implies $h_* = t = 0$.

Definition 3.2. We call a pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$ in \mathcal{G} a **G -exact sequence** if $0 \rightarrow \text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{G}}(G, C) \rightarrow 0$ is a short exact sequence in the category of abelian groups (so also in $\text{Mod-}R$). We often denote a G -exact sequence by $A \rightarrowtail B \twoheadrightarrow C$, and call $A \rightarrowtail B$ a **G -monomorphism** and $B \twoheadrightarrow C$ a **G -epimorphism**. We will also call a subobject $P \subseteq A$ a **G -subobject** if the inclusion map is a G -monomorphism, and denote this $P \subseteq_G A$.

We list some basic properties of G -exact sequences.

Proposition 3.3. *We have the following properties of G -exact sequences.*

- (1) *Any G -exact sequence is an exact sequence in \mathcal{G} .*
- (2) *The class of all G -exact sequences is closed under isomorphisms and contains all split exact sequence.*
- (3) *A pushout of a G -monomorphism is again a G -monomorphism. In fact, $\text{Hom}_{\mathcal{G}}(G, -)$ takes pushouts of G -monomorphisms to pushouts in $\text{Mod-}R$. We also have that pullbacks of G -epimorphisms are again G -epimorphisms. Moreover, $\text{Hom}_{\mathcal{G}}(G, -)$ takes all pullbacks in \mathcal{G} to pullbacks in $\text{Mod-}R$ since it is a right adjoint.*
- (4) *G -monomorphisms are closed under composition and G -epimorphisms are closed under composition.*

Proof. For (1), note that in the definition of G -exact sequence we have $0 = g_* f_* = (g f)_*$. So $\text{Hom}_{\mathcal{G}}(G, -)$ faithful implies $0 = g f$. So we can view $A \xrightarrow{f} B \xrightarrow{g} C$ as a chain complex in \mathcal{G} , and so (1) follows from Lemma 3.1.

(2) is clear.

For (3), we first show that a pullback of a G -epimorphism is a G -epimorphism. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a given G -exact sequence. Taking a pullback $B \xrightarrow{g} C \leftarrow X$ leads to a diagram of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f'} & P & \xrightarrow{g'} & X & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Applying $\text{Hom}_{\mathcal{G}}(G, -)$ to this diagram gives us a commutative diagram with the bottom row exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, A) & \xrightarrow{f'_*} & \text{Hom}_{\mathcal{G}}(G, P) & \xrightarrow{g'_*} & \text{Hom}_{\mathcal{G}}(G, X) & & \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, A) & \xrightarrow{f_*} & \text{Hom}_{\mathcal{G}}(G, B) & \xrightarrow{g_*} & \text{Hom}_{\mathcal{G}}(G, C) & \longrightarrow & 0 \end{array}$$

But the functor $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \rightarrow \text{Mod-}R$ is a right adjoint and so it preserves limits, so in particular it preserves pullbacks. Therefore the right square is a pullback in $\text{Mod-}R$. So since g_* is an epimorphism we get that g'_* must also be an epimorphism. This proves $0 \rightarrow A \xrightarrow{f'} P \xrightarrow{g'} X \rightarrow 0$ is a G -exact sequence.

Next, we wish to show that a pushout of a G -monomorphism is a G -monomorphism.

So consider a G -exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$. Taking a pushout of $X \leftarrow A \xrightarrow{f} B$ leads to a diagram of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & X & \xrightarrow{f'} & P & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

We only need to show that g'_* is an epimorphism. Since $\text{Hom}_{\mathcal{G}}(G, -)$ is not a left adjoint we can't expect it to preserve all pushouts. However, note that since $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \rightarrow \mathcal{S}$ is an equivalence it takes pushouts in \mathcal{G} to pushouts in the abelian category \mathcal{S} . This implies that we get the \mathcal{S} -diagram below with \mathcal{S} -exact rows and with the left square being a pushout in \mathcal{S} .

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, A) & \xrightarrow{f_*} & \text{Hom}_{\mathcal{G}}(G, B) & \xrightarrow{g_*} & \text{Hom}_{\mathcal{G}}(G, C) & & \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, X) & \xrightarrow{f'_*} & \text{Hom}_{\mathcal{G}}(G, P) & \xrightarrow{g'_*} & \text{Hom}_{\mathcal{G}}(G, C) & & \end{array}$$

But by hypothesis, g_* is an epimorphism in $\text{Mod-}R$, and so we see immediately that g'_* is also an epimorphism in $\text{Mod-}R$. This shows that $X \xrightarrow{f'} P \xrightarrow{g'} C$ is a G -exact sequence. In fact, since the rows of the diagram above are exact in $\text{Mod-}R$, it follows that the left hand square is actually the pushout in $\text{Mod-}R$. So the functor $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \rightarrow \text{Mod-}R$ preserves pushouts of G -monomorphisms.

For (4), we first show that G -epimorphisms are closed under composition. Say $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are each G -epimorphisms. Since each is an epimorphism, so is

the composition hg . Then $0 \rightarrow \ker hg \rightarrow B \xrightarrow{hg} D \rightarrow 0$ must be a G -exact sequence since $(hg)_* = h_*g_*$ is an epimorphism.

Finally, we wish to show that G -monomorphisms are closed under composition. So let $A \xrightarrow{i} B$ and $B \xrightarrow{j} C$ each be G -monomorphisms. Taking the pushout of $B/A \leftarrow B \xrightarrow{j} C$ leads to a diagram of short exact sequences.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & A & \xlongequal{\quad} & A & & \\
& & i \downarrow & & ji \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{j} & C & \xrightarrow{\pi} & C/B \longrightarrow 0 \\
& & \downarrow & & g \downarrow & & \parallel \\
0 & \longrightarrow & B/A & \xrightarrow{j'} & P & \xrightarrow{\pi'} & C/B \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Since the row $0 \rightarrow B \xrightarrow{j} C \xrightarrow{\pi} C/B \rightarrow 0$ is G -exact, we have by what was proved already that the pushout row $0 \rightarrow B/A \xrightarrow{j'} P \xrightarrow{\pi'} C/B \rightarrow 0$ must also be G -exact. So applying $\text{Hom}_{\mathcal{G}}(G, -)$ yields a commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \text{Hom}_{\mathcal{G}}(G, A) & \xlongequal{\quad} & \text{Hom}_{\mathcal{G}}(G, A) & & \\
& & i_* \downarrow & & (ji)_* \downarrow & & \\
0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, B) & \xrightarrow{j_*} & \text{Hom}_{\mathcal{G}}(G, C) & \xrightarrow{\pi_*} & \text{Hom}_{\mathcal{G}}(G, C/B) \longrightarrow 0 \\
& & \downarrow & & g_* \downarrow & & \parallel \\
0 & \longrightarrow & \text{Hom}_{\mathcal{G}}(G, B/A) & \xrightarrow{j'_*} & \text{Hom}_{\mathcal{G}}(G, P) & \xrightarrow{\pi'_*} & \text{Hom}_{\mathcal{G}}(G, C/B) \longrightarrow 0 \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

We are trying to show that g_* is an epimorphism in $\text{Mod-}R$, and now the snake lemma shows that it is. \square

We show in Appendix B that when working in abelian categories, Quillen's notion of an *exact category* from [Qui73] coincides with the notion of a *proper class of short exact sequences* from [Mac63, Chapter XII.4].

Corollary 3.4. *Let \mathcal{G} be a Grothendieck category with generator G . Let \mathcal{E} denote the class of all G -exact sequences. Then $(\mathcal{G}, \mathcal{E})$ is an exact category. Equivalently, \mathcal{E} is a proper class of short exact sequences. We will let $\mathcal{G}_G = (\mathcal{G}, \mathcal{E})$ denote this exact*

category and we will call it the **G -exact category** of \mathcal{G} . The functor $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G}_G \rightarrow \text{Mod-}R$ is exact.

Proof. The four properties of Proposition 3.3 are the axioms of an exact category in [Büh10]. It is clear from definitions that the functor $\text{Hom}_{\mathcal{G}}(G, -) : \mathcal{G}_G \rightarrow \text{Mod-}R$ is exact. We refer the reader to Appendix B for the equivalence with proper classes. \square

The generator $G = \oplus_{i \in I} G_i$ is not just a generator for \mathcal{G} . It is easy to see that it is also a generator for \mathcal{G}_G , but we first explain what we mean by this.

In [Hov02], Hovey worked with abelian categories along with a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4]. There he defined an object U to be a generator for a proper class \mathcal{P} if for all maps f , $\text{Hom}_{\mathcal{G}}(U, f)$ surjective implies f is a \mathcal{P} -epimorphism. Also here, a set $\{U_i\}$ generates \mathcal{P} if $U = \oplus U_i$ is a generator for \mathcal{P} . On the other hand, in [SŠ11] and [Sto13], the authors work with exact categories and define a set $\{U_i\}$ to be generating if for any object A , there is an admissible epimorphism $\pi : U \twoheadrightarrow A$ where U is some set-indexed direct sum of objects from $\{U_i\}$. The following corollary shows that G is a generator for \mathcal{G}_G in both senses. We therefore can feel free to reference the above authors' results.

Corollary 3.5. $G = \oplus_{i \in I} G_i$ is a projective generator for the G -exact category \mathcal{G}_G . In particular, the following hold:

- (1) By definition, an object P is projective in \mathcal{G}_G if the functor $\text{Hom}_{\mathcal{G}}(P, -)$ takes G -exact sequences to short exact sequences. We will call such an object **G -projective**. Notice that the construction of the G -exact category immediately forces G and each G_i to be G -projective.
- (2) G is a generator for \mathcal{G}_G . That is, if $\text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B)$ is surjective, then f is a G -epimorphism.
Equivalently, $\{G_i\}$ is a set of generators for \mathcal{G}_G . That is, if $\text{Hom}_{\mathcal{G}}(G_i, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G_i, B)$ is surjective for all G_i , then f is a G -epimorphism.
- (3) \mathcal{G}_G has enough projectives. In particular, for each $A \in \mathcal{G}$, we can find a G -epimorphism $\oplus_{i \in I} G \twoheadrightarrow A$. Equivalently, we can find a G -epimorphism $X \twoheadrightarrow A$ where X is a direct sum of copies of some of the G_i .
- (4) An object P is G -projective if and only if it is a direct summand of a direct sum of copies of some of the G_i .

Proof. For (2), let $f : A \rightarrow B$ be such that $\text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B)$ is surjective. Since G is a generator for \mathcal{G} this implies f is an epimorphism and so there is a short exact sequence $0 \rightarrow K \rightarrow A \xrightarrow{f} B \rightarrow 0$. By definition this sequence is G -exact, so we are done. In terms of the generating set $\{G_i\}$, just note that $\text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B)$ is surjective iff $\text{Hom}_{\mathcal{G}}(G_i, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G_i, B)$ is surjective for all G_i .

For (3), in the usual way, take $I = \text{Hom}_{\mathcal{G}}(G, A)$, and define $\oplus_{t \in I} G \twoheadrightarrow A$ in component $(t : G \rightarrow A) \in I$ to be t itself. It is immediate that this is a G -epimorphism. $\oplus_{t \in I} G$ is indeed a G -projective object, since in any exact category, direct sums of projectives are again projectives by [Büh10, Corollary 11.7]. For (4), we see that the G -epimorphism $\oplus_{t \in I} G \twoheadrightarrow P$ splits if and only if P is G -projective by [Büh10, Corollary 11.6]. \square

3.2. G -subobjects. Here we go on to list more properties of G -monomorphisms, but we state them in terms of G -subobjects. This is the form in which we will use them later. Note that they are analogous to properties of pure submodules. Recall that we write $P \subseteq_G A$ to mean that P is a G -subobject of A , that is, $\text{Hom}_G(G, A) \rightarrow \text{Hom}_G(G, A/P)$ is surjective.

Proposition 3.6. *Consider subobject $A \subseteq B \subseteq C$ in \mathcal{G} .*

- (1) *If $A \subseteq_G B$ and $B \subseteq_G C$ then $A \subseteq_G C$.*
- (2) *If $A \subseteq_G C$ then $A \subseteq_G B$.*
- (3) *If $A \subseteq_G C$ and $B/A \subseteq_G C/A$ then $B \subseteq_G C$.*

Proof. (1) has already appeared as part (4) of Proposition 3.3. (2) follows from general facts about admissible monomorphisms in (weakly idempotent complete) exact categories. See [Büh10, Prop. 7.6 or Prop. 2.16].

For (3), all we need to check is that the map $\text{Hom}_G(G, C) \rightarrow \text{Hom}_G(G, C/B)$ is an epimorphism. But this is just the composite

$$\text{Hom}_G(G, C) \rightarrow \text{Hom}_G(G, C/A) \rightarrow \text{Hom}_G(G, (C/A)/(B/A)) \cong \text{Hom}_G(G, C/B),$$

and these are epimorphisms by hypothesis. \square

4. THE G -DERIVED CATEGORY

Again let \mathcal{G} be a Grothendieck category and let $G = \oplus G_i$ where $\{G_i\}$ is a set of generators. In this section we construct the derived category $\mathcal{D}(G)$. It is the derived category of the G -exact category \mathcal{G}_G and we obtain it by putting a suitable model structure on $\text{Ch}(\mathcal{G})$. Following the general definition of an exact chain complex from [Büh10, Definition 10.1], the exact complexes in \mathcal{G}_G are the G -acyclic complexes. That is, those chain complexes X for which $\text{Hom}_G(G, X)$ is exact. So we wish to “kill” these complexes by making them the trivial objects of an exact model structure.

4.1. The category $\text{Ch}(\mathcal{G})_G$. Our convention when working with chain complexes is that the differential lowers degree, so $\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots$ is a chain complex. Given $X \in \text{Ch}(\mathcal{G})$, the n th suspension of X , denoted $\Sigma^n X$, is the complex given by $(\Sigma^n X)_k = X_{k-n}$ and $(d_{\Sigma^n X})_k = (-1)^n d_{k-n}$. Given two chain complexes X and Y we define $\text{Hom}(X, Y)$ to be the complex of abelian groups $\cdots \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n-1}) \rightarrow \cdots$, where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. This gives a functor $\text{Hom}(X, -): \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathbb{Z})$. Note that this functor takes exact sequences to left exact sequences, and it is exact if each X_n is projective. Similarly the contravariant functor $\text{Hom}(-, Y)$ sends exact sequences to left exact sequences and is exact if each Y_n is injective. It is an exercise to check that the homology satisfies $H_n[\text{Hom}(X, Y)] = \text{Ch}(\mathcal{G})(X, \Sigma^{-n} Y) / \sim$ where \sim is the usual relation of chain homotopic maps.

For a given $A \in \mathcal{G}$, we denote the n -disk on A by $D^n(A)$. This is the complex consisting only of $A \xrightarrow{1_A} A$ concentrated in degrees n and $n-1$. We denote the n -sphere on A by $S^n(A)$, and this is the complex consisting of A in degree n and 0 elsewhere.

Recall that \mathcal{G}_G is the same category as \mathcal{G} , with the same morphisms, but with an exact structure coming from the proper class of G -exact sequences. In the same way, we let $\text{Ch}(\mathcal{G})_G$ denote the category of all chain complexes, with the usual

chain maps, but considered as an exact category where the short exact sequences are G -exact in each degree. We will call these *degreewise G -exact sequences*. It is indeed a general fact that for any exact category $\mathcal{A} = (\mathcal{A}, \mathcal{E})$, the category $\text{Ch}(\mathcal{A})$ becomes an exact category when considered along with the short exact sequences which degreewise lie in \mathcal{E} . So one might argue that the proper notation in our case is $\text{Ch}(\mathcal{G}_G)$, rather than $\text{Ch}(\mathcal{G})_G$. However, we have the following lemma.

Lemma 4.1. *Consider the standard generating set $\{D^n(G_i)\}$ in $\text{Ch}(\mathcal{G})$ and let $G = \bigoplus D^n(G_i)$ be the direct sum, taken over all $n \in \mathbb{Z}$ and $i \in I$. Then the G -exact category $\text{Ch}(\mathcal{G})_G$ of Corollary 3.4 coincides with $\text{Ch}(\mathcal{G}_G)$. That is, the proper class of G -exact sequences in $\text{Ch}(\mathcal{G})$ (here $G = \bigoplus D^n(G_i)$) coincides with the class of all short exact sequences which degreewise are G -exact sequences (here $G = \bigoplus G_i$) in \mathcal{G} .*

Proof. Consider a short sequence $X \rightarrowtail Y \twoheadrightarrow Z$ of complexes. Then it is G -exact iff

$$\text{Hom}_{\text{Ch}(\mathcal{G})}(G, X) \rightarrowtail \text{Hom}_{\text{Ch}(\mathcal{G})}(G, Y) \twoheadrightarrow \text{Hom}_{\text{Ch}(\mathcal{G})}(G, Z)$$

is a short exact sequence of abelian groups, iff

$$\prod_{n,i} \text{Hom}(D^n(G_i), X) \rightarrowtail \prod_{n,i} \text{Hom}(D^n(G_i), Y) \twoheadrightarrow \prod_{n,i} \text{Hom}(D^n(G_i), Z)$$

is short exact, iff $\prod_{n,i} \text{Hom}_G(G_i, X_n) \rightarrowtail \prod_{n,i} \text{Hom}_G(G_i, Y_n) \twoheadrightarrow \prod_{n,i} \text{Hom}_G(G_i, Z_n)$ is short exact, iff $X \rightarrowtail Y \twoheadrightarrow Z$ is degreewise G -exact (where here $G = \bigoplus G_i$) in \mathcal{G} . \square

Being an exact category, $\text{Ch}(\mathcal{G})_G$ comes with a Yoneda Ext group, which in this case is the group of (equivalence classes of) degreewise G -exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$, with addition defined by the Baer sum. We will denote this bifunctor by $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1$, and note that for given chain complexes X and Y , $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(X, Y)$ is a subgroup of the usual Yoneda $\text{Ext}_{\text{Ch}(\mathcal{G})}^1(X, Y)$. We sometimes will also call an element of $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(X, Y)$ a **degreewise G -extension**. We also denote by $\text{G-Ext}_{\mathcal{G}}^1$, the group of **G -extensions** in the ground category \mathcal{G}_G . We have the following G -versions of standard isomorphisms.

Lemma 4.2. *Let $A \in \mathcal{G}$ and $X \in \text{Ch}(\mathcal{G})$. Then we have the following natural isomorphisms.*

- (1) $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(D^n(A), X) \cong \text{G-Ext}_{\mathcal{G}}^1(A, X_n)$
- (2) $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(X, D^{n+1}(A)) \cong \text{G-Ext}_{\mathcal{G}}^1(X_n, A)$

Proof. The point is that the standard isomorphisms take degreewise G -extensions to G -extensions. For example, for (1), the standard mapping $\text{Ext}_{\text{Ch}(\mathcal{G})}^1(D^n(A), X) \rightarrow \text{Ext}_{\mathcal{G}}^1(A, X_n)$ takes a short exact sequence $0 \rightarrow X \rightarrow Z \rightarrow D^n(A) \rightarrow 0$ to $0 \rightarrow X_n \rightarrow Z_n \rightarrow A \rightarrow 0$. Its inverse is formed by taking an extension $0 \rightarrow X_n \rightarrow Z \rightarrow A \rightarrow 0$ and forming the pushout of $X_{n-1} \xleftarrow{d_n} X_n \rightarrow Z$. Since pushouts of G -monomorphisms are again G -monomorphisms, we see that the isomorphisms restrict nicely between G -extensions. This shows (1). The isomorphism (2) is dual, using that pullbacks of G -epimorphisms are again G -epimorphisms. \square

There is one more exact category that will be of use. We denote by $\text{Ch}(\mathcal{G})_{dw}$ the category of all chain complexes along with the proper class of all degreewise split

short exact sequences. We denote its Yoneda Ext bifunctor by Ext_{dw}^1 . We note that we have subgroup containments

$$\text{Ext}_{dw}^1(X, Y) \subseteq \text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(X, Y) \subseteq \text{Ext}_{\text{Ch}(\mathcal{G})}^1(X, Y),$$

and we have the following well-known connection between Ext_{dw}^1 and the functor Hom .

Lemma 4.3. *For chain complexes X and Y , we have isomorphisms:*

$$\text{Ext}_{dw}^1(X, \Sigma^{(-n-1)}Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(\mathcal{G})(X, \Sigma^{-n}Y) / \sim$$

In particular, for chain complexes X and Y , $\text{Hom}(X, Y)$ is exact iff for any $n \in \mathbb{Z}$, any chain map $f: \Sigma^n X \rightarrow Y$ is homotopic to 0 (or iff any chain map $f: X \rightarrow \Sigma^n Y$ is homotopic to 0).

We note also that the functor $\text{Hom}(X, -): \text{Ch}(\mathcal{G}) \rightarrow \text{Ch}(\mathbb{Z})$ takes degreewise G -exact sequences to short exact sequences if each X_n is G -projective. Similarly the contravariant functor $\text{Hom}(-, Y)$ sends degreewise G -exact sequences to short exact sequences if each Y_n is G -injective.

4.2. G -acyclic complexes. Following definition [Büh10, Definition 10.1]), an acyclic chain complex with respect to the exact structure \mathcal{G}_G ought to be a chain complex X for which its differentials each factor as $X_n \twoheadrightarrow Z_{n-1}X \hookrightarrow X_{n-1}$ in such a way that $Z_nX \hookrightarrow X_n \twoheadrightarrow Z_{n-1}X$ is G -acyclic. We will call such a complex G -acyclic (or G -exact).

Lemma 4.4. *We have the following properties of G -acyclic complexes.*

- (1) *Let X be a chain complex. Then the following are equivalent:*
 - (a) *X is G -acyclic.*
 - (b) *X is exact and $Z_nX \subseteq_G X_n$ is a G -subobject for each n .*
 - (c) *$\text{Hom}_{\mathcal{G}}(G, X)$ is exact.*
 - (d) *Each $\text{Hom}_{\mathcal{G}}(G_i, X)$ is exact.**Note in particular that any G -acyclic complex is exact in the usual sense.*
- (2) *If X is contractible, meaning $1_X \sim 0$, then X is G -acyclic.*
- (3) *The class of G -acyclic complexes is thick in $\text{Ch}(\mathcal{G})_G$. That is, it is closed under retracts and for any exact $X \twoheadrightarrow Y \twoheadrightarrow Z$ in $\text{Ch}(\mathcal{G})_G$, if two out of three terms are G -acyclic then so is the third.*

Proof. For (1), we clearly have (a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d). (d) and (c) are equivalent because $\text{Hom}_{\mathcal{G}}(G, X) \cong \prod_{i \in I} \text{Hom}_{\mathcal{G}}(G_i, X)$ (and a product of exact complexes is exact in **Ab**). Using Lemma 3.1 we see (c) implies (b).

For (2), recall that having $1_X \sim 0$ means there exists maps $\{s_n: X_n \rightarrow X_{n+1}\}$ such that $sd + ds = 1$. Applying the additive functor $\text{Hom}_{\mathcal{G}}(G, -)$ to this equation shows that $\text{Hom}_{\mathcal{G}}(G, X)$ is also contractible. In particular it is exact.

For (3), note that if $X \hookrightarrow Y \twoheadrightarrow Z$ is a short exact sequence in $\text{Ch}(\mathcal{G})_G$, then since it is degreewise G -exact we get a short exact sequence of complexes of abelian groups $0 \rightarrow \text{Hom}_{\mathcal{G}}(G, X) \rightarrow \text{Hom}_{\mathcal{G}}(G, Y) \rightarrow \text{Hom}_{\mathcal{G}}(G, Z) \rightarrow 0$. If any two out of three of these are exact then so is the third. For retracts, note that any additive functor preserves retracts. So this is true since a retract of an exact complex of abelian groups is again an exact complex. \square

4.3. Projectives in $\text{Ch}(\mathcal{G})_G$. Here we classify the projective objects of $\text{Ch}(\mathcal{G})_G$.

Lemma 4.5. *Call a chain complex X in $\text{Ch}(\mathcal{G})$ a **G -projective** complex if it is projective in the exact category $\text{Ch}(\mathcal{G})_G$. The following are equivalent:*

- (1) X is G -projective.
- (2) X is G -acyclic with each $Z_n X$ a G -projective.
- (3) X is isomorphic to a split exact complex with G -projective components.
That is, $X \cong \oplus_{n \in \mathbb{Z}} D^n(P_n)$ where each P_n is a G -projective.
- (4) X is a contractible complex with each X_n G -projective.

Proof. Using part (3) of Corollary 3.5 and [Gil13, Corollary 2.7] we can find, for any chain complex X , a G -epimorphism $\oplus_{n \in \mathbb{Z}} D^n(P_n) \rightarrow X$ in which each P_n is G -projective. If X is G -projective, then this is a split epi. Then (2),(3), and (4) all follow and are equivalent by standard arguments. On the other hand, the isomorphism $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1(D^n(A), X) \cong G\text{-Ext}_{\mathcal{G}}^1(A, X_n)$ of Lemma 4.2 tells us that a disk $D^n(A)$ is G -projective if and only if A is G -projective in \mathcal{G}_G . Moreover, in any exact category a direct sum is projective if and only if each summand is projective by [Büh10, Corollary 11.7]. \square

4.4. The G -derived category. We now construct the G -derived category by putting a cofibrantly generated “projective” model structure on $\text{Ch}(\mathcal{G})_G$. The model structure follows as a Corollary to the next theorem. The proof relies on Quillen’s small object argument. We refer to the version in [Hov99, Theorem 2.1.14] and in particular we refer the reader there for the definition of the notation I -cell and I -inj. We also refer the reader to [Hov02] for the language of cotorsion pairs.

We define two sets of maps which will respectively be the *generating cofibrations* and *generating trivial cofibrations*:

$$I = \{0 \rightarrow D^n(G_i)\} \cup \{S^{n-1}(G_i) \rightarrow D^n(G_i)\}, \quad \text{and} \quad J = \{0 \rightarrow D^n(G_i)\}.$$

We also define the following set of objects which will cogenerate the cotorsion pair:

$$\mathcal{S} = \{D^n(G_i)\} \cup \{S^n(G_i)\}.$$

Note that $\mathcal{S} = \text{cok } I = \{\text{cok } i \mid i \in I\}$. We leave it to the reader to check the easy fact that a chain complex X satisfies $X \in \mathcal{S}^\perp$ if and only if $(X \rightarrow 0) \in I\text{-inj}$. (The “perp” here is taken with respect to the degree-wise G -exact sequences. So use $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1$ and the fact that the $D^n(G_i)$ are G -projective.)

Theorem 4.6. *Let \mathcal{G} be any Grothendieck category with a generator $G = \oplus G_i$. Let \mathcal{W} denote the class of all G -acyclic complexes. Then the set $\mathcal{S} = \{D^n(G_i)\} \cup \{S^n(G_i)\}$ cogenerates a cotorsion pair $(\mathcal{P}, \mathcal{W})$ in the exact category $\text{Ch}(\mathcal{G})_G$ with the following properties.*

- (1) $(\mathcal{P}, \mathcal{W})$ is complete. In fact, for any chain complex X there is a G -exact sequence $W \rightarrow P \rightarrow X$ where $W \in \mathcal{W}$ and $P \in \mathcal{P}$ is a transfinite (degree-wise-split) extension of \mathcal{S} . In particular, each P_n is a direct sum of copies of the G_i .
- (2) $P \in \mathcal{P}$ if and only if P is a retract of a transfinite (degree-wise-split) extension of \mathcal{S} . We will call a complex in \mathcal{P} a **semi- G -projective** complex.
- (3) \mathcal{W} is thick and $\mathcal{P} \cap \mathcal{W}$ coincides with the class of projective complexes in $\text{Ch}(\mathcal{G})_G$. (See Lemma 4.5.)

For $\mathcal{G} = R\text{-Mod}$ and $G = R$, this recovers the usual projective model structure on $\text{Ch}(R)$ where the cofibrant complexes are the DG-projective complexes. Some authors call these complexes *semiprojective*, and since DG- G -projective looks odd we use semiprojective.

Our proof of Theorem 4.6 is based on the proof of [Hov02, Theorem 6.5]. Indeed for the case when \mathcal{G} is locally finitely presentable (that is, $G = \bigoplus G_i$ where the G_i are finitely presented), we only need the first paragraph of the proof below, combined with Corollary 5.3 and [Hov02, Theorem 6.5].

Proof. Since each G_i is G -projective we have an equality

$$\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(S^n(G_i), X) = \text{Ext}_{dw}^1(S^n(G_i), X).$$

So $X \in \{S^n(G_i)\}^\perp$ if and only if for each n we have vanishing of

$$\text{Ext}_{dw}^1(S^n(G_i), X) = H_{n-1} \text{Hom}(S^0(G_i), X) = H_{n-1} \text{Hom}_{\mathcal{G}}(G_i, X).$$

So $X \in \{S^n(G_i)\}^\perp$ if and only if X is G -acyclic. So indeed \mathcal{S} cogenerates a cotorsion pair $(\mathcal{P}, \mathcal{W})$ in the exact category $\text{Ch}(\mathcal{G})_G$.

To show this cotorsion pair is complete we apply the small object argument from [Hov99, Theorem 2.1.14]. We can do this since every object in a Grothendieck category is *small*. The small object argument provides, for a given map $X \rightarrow Y$, a functorial factorization $X \rightarrow Z \rightarrow Y$ where $(X \rightarrow Z) \in I\text{-cell}$ and $(Z \rightarrow Y) \in I\text{-inj}$. So we now pause to better understand $I\text{-cell}$ and $I\text{-inj}$.

Claim: If $(p : Z \rightarrow Y) \in I\text{-inj}$, then p is a degreewise G -epimorphism with G -acyclic kernel. For completeness, we include the following direct proof of the claim. However, we note that $I\text{-inj}$ can also be characterized by applying [Hov01, Propositions 1.3–1.6]. Indeed take the set \mathcal{M} in [Hov01, Definition 1.1] to be $\mathcal{M} = \{0 \rightarrowtail G_i\} \cup \{0 \rightarrowtail 0\}$.

To prove the claim, say we have such a $p : Z \rightarrow Y$ in $I\text{-inj}$. Then for each n and i we have a lift in the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ D^n(G_i) & \longrightarrow & Y \end{array}$$

This implies that each p_n is a G -epimorphism. So now we have $K \rightarrowtail Z \twoheadrightarrow Y$ is G -exact where $K = \ker p$. It is left to show K is G -acyclic. For *any* set of maps I , it is an easy exercise to check that $I\text{-inj}$ is closed under pullbacks. Since $K \rightarrow 0$ lies in the pullback square

$$\begin{array}{ccc} K & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & Y \end{array}$$

we see $(K \rightarrow 0) \in I\text{-inj}$. But as pointed out above the statement of the theorem, this is equivalent to saying $K \in \mathcal{S}^\perp$. So K is G -acyclic.

Claim: If $(f : X \rightarrow Z) \in I\text{-cell}$, then f is a degreewise split monomorphism with cokernel a transfinite extension of \mathcal{S} .

To prove this, say $f : X \rightarrow Z$ is in $I\text{-cell}$. By definition, f is a transfinite composition of pushouts of maps of the form $0 \rightarrowtail D^n(G_i)$ or $S^{n-1}(G_i) \rightarrowtail D^n(G_i)$.

Note that such pushouts are necessarily degree-wise split monomorphisms whose cokernels are in \mathcal{S} . This means $(f : X \rightarrow Z) = (X_0 \xrightarrow{f} \varinjlim_{\alpha < \lambda} X_\alpha)$ is a transfinite (degree-wise split) extension of $X = X_0$ by \mathcal{S} . Since transfinite extensions of split monomorphisms are again split monomorphisms, we conclude that f too is a degree-wise split monomorphism. We now look at $\text{cok } f$. Since direct limits are exact we have a short exact sequence $X_0 \rightarrow \varinjlim X_\alpha \rightarrow \varinjlim (X_\alpha/X_0)$. In particular, $\text{cok } f \cong \varinjlim (X_\alpha/X_0)$ is a transfinite extension of

$$0 \rightarrow X_1/X_0 \rightarrow X_2/X_0 \rightarrow X_3/X_0 \rightarrow \cdots \rightarrow X_\alpha/X_0 \rightarrow \cdots$$

But $(X_{\alpha+1}/X_0)/(X_\alpha/X_0) \cong X_{\alpha+1}/X_\alpha \in \mathcal{S}$. This proves f is a degree-wise split monomorphism with cokernel a transfinite extension of \mathcal{S} .

We now can prove that $(\mathcal{P}, \mathcal{W})$ is complete. So suppose Y is an arbitrary chain complex and use the small object argument to factor $0 \rightarrow Y$ as $0 \rightarrow Z \xrightarrow{p} Y$ where $0 \rightarrow Z \in I\text{-cell}$ and $Z \xrightarrow{p} Y$ is in $I\text{-inj}$. Then $K \rightarrow Z \rightarrow Y$ is a degree-wise G -exact sequence with K a G -acyclic complex. Also, Z must be a transfinite extension of \mathcal{S} . But by [Hov02, Lemma 6.2] (taking the G -exact sequences as the *proper class* of short exact sequences) we have that \mathcal{P} is closed under retracts and transfinite extensions. Therefore $Z \in \mathcal{P}$ and $(\mathcal{P}, \mathcal{W})$ has enough projectives in the way we claim in (1). To see that $(\mathcal{P}, \mathcal{W})$ has enough injectives we instead factor $X \rightarrow 0$ as $X \xrightarrow{f} Z \rightarrow 0$ where $X \xrightarrow{f} Z \in I\text{-cell}$ and $Z \rightarrow 0$ is in $I\text{-inj}$. Then f is a degree-wise split monomorphism (so a G -mono) with $\text{cok } f \in \mathcal{P}$, and $Z \in \mathcal{W}$.

Next, statement (2). As mentioned above, \mathcal{P} is closed under retracts. Statement (2) is then a result of the following observation: Given $Q \in \mathcal{P}$, write a G -exact sequence $W \rightarrow P \rightarrow Q$ where $W \in \mathcal{W}$ and $P \in \mathcal{P}$ is a transfinite (degree-wise-split) extension of \mathcal{S} . This G -exact sequence is an element of $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1(Q, W) = 0$. So it splits and Q is a retract of P as desired.

For (3), we see from Lemma 4.4 that \mathcal{W} is thick and contains all contractible complexes. So in particular \mathcal{W} contains the projective objects of $\text{Ch}(\mathcal{G})_G$ by Lemma 4.5. Since $(\mathcal{P}, \mathcal{W})$ is complete with \mathcal{W} thick and containing the projectives, the result follows by the argument in [BGH13, Proposition 3.4]. \square

In the language of [Gil11] and [Gil12], parts (1) and (3) of the above Theorem say that $(\mathcal{P}, \mathcal{W})$ is a projective cotorsion pair in $\text{Ch}(\mathcal{G})_G$. Such a cotorsion pair is equivalent to a model structure for which every object is fibrant. The following corollary records some basic facts about this model structure.

Corollary 4.7. *Let \mathcal{G} be any Grothendieck category with a generator $G = \oplus G_i$. Then there is a model structure on $\text{Ch}(\mathcal{G})$ which we call the **G -projective model structure** whose trivial objects are the G -acyclic complexes. This gives us a model for the **G -derived category**, which we denote by $\mathcal{D}(G)$. The model structure satisfies the following:*

- (1) *The fibrations are precisely the G -epimorphisms. That is, the chain maps which are G -epimorphisms in each degree.*
- (2) *The trivial fibrations are the G -epimorphisms with G -acyclic kernel.*
- (3) *The cofibrations are the degree-wise split monomorphisms whose cokernel is a semi- G -projective complex.*
- (4) *The trivial cofibrations are the split monomorphisms whose cokernel is a G -projective complex.*

- (5) The weak equivalences are the ***G*-homology isomorphisms**. That is, the chain maps $f : X \rightarrow Y$ for which $\text{Hom}_{\mathcal{G}}(G, f) : \text{Hom}_{\mathcal{G}}(G, X) \rightarrow \text{Hom}_{\mathcal{G}}(G, Y)$ is a homology isomorphism.
- (6) The model structure is cofibrantly generated. The sets I and J from above are respectively the generating cofibrations and generating trivial cofibrations. Thus $\mathcal{D}(G)$ is well generated in the sense of [Nee01].
- (7) If each G_i is finitely presented, then the model structure is finitely generated and so in this case $\mathcal{D}(G)$ is compactly generated.

Proof. In the exact category $\text{Ch}(\mathcal{G})_G$, we have the complete cotorsion pair $(\mathcal{P}, \mathcal{W})$. We also have the complete cotorsion pair $(\mathcal{Q}, \mathcal{A})$ where \mathcal{Q} is the class of G -projective complexes of Lemma 4.5 and \mathcal{A} is the class of all complexes. Theorem 4.6 along with the main theorem of [Hov02] imply that we automatically have the model structure with (trivial) fibrations and (trivial) cofibrations as described.

In the correspondence between cotorsion pairs and model structures, the weak equivalences are precisely the maps which factor as a trivial cofibration followed by a trivial fibration. We wish to see that such maps are exactly the G -homology isomorphisms. First, given any $f : X \rightarrow Y$, let's denote the composite functor $H_n[\text{Hom}_{\mathcal{G}}(G, f)]$ simply by $H_n f_*$. Using the model structure we can apply the factorization axiom and write $f = pi$ where p is a fibration and i is a trivial cofibration. We have $H_n f_* = H_n p_* \circ H_n i_*$. Since i is a split monomorphism with G -projective (so G -acyclic) cokernel, we see $H_n i_*$ is an isomorphism. So $H_n f_*$ is an isomorphism if and only if $H_n p_*$ is an isomorphism. Since p is a G -epimorphism, we see $H_n p_*$ is an isomorphism (for all n) if and only if $\ker p$ is G -acyclic. That is, iff p is a trivial fibration. We have now shown that f factors as a trivial cofibration followed by a trivial fibration iff $H_n f_*$ is an isomorphism for all n .

It is easy to see that $J\text{-inj}$ is the class of G -epimorphisms. This means that J is the set of generating trivial cofibrations. We also showed that everything in $I\text{-inj}$ is a G -epimorphism with G -acyclic kernel. So it is left to show that every G -epimorphism with G -acyclic kernel is in $I\text{-inj}$. So let $X \xrightarrow{p} Y$ be a G -epimorphism with kernel $K \in \mathcal{W}$. Being a G -epimorphism we know that there is a lift in any diagram of the form

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow p \\ D^n(G_i) & \longrightarrow & Y \end{array}$$

So all we need to show is that there is a lift for any diagram

$$\begin{array}{ccc} S^{n-1}(G_i) & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ D^n(G_i) & \xrightarrow{g} & Y \end{array}$$

But again, we may start by finding an $D^n(G_i) \xrightarrow{h} X$ such that $ph = g$. We check that $(f - hi)$ lands in the kernel K . Now since $\text{G-Ext}_{\mathcal{G}}^1(S^{n-1}(G_i), K) = 0$, we see that the map $(f - hi)$ extends to some $D^n(G_i) \xrightarrow{\psi} K$. That is, $\psi i = (f - hi)$. So now we check that $(h + j\psi)$, where $j : K \rightarrow X$ is the desired lift. (i) $p(h + j\psi) = ph + 0 = ph = g$. (ii) $(h + j\psi)i = hi + j\psi i = hi + (f - hi) = f$.

Since we have a cofibrantly generated model structure on a locally presentable (pointed) category, a main result from [Ros05] assures us that $\mathcal{D}(G) = \text{Ho}(\text{Ch}(\mathcal{G}))$ is well generated in the sense of [Nee01] and [Kra01]. In the case that $G = \bigoplus G_i$ has each G_i finitely presented, then the G_i are *finite* in the sense of [Hov99, Section 7.4]. We then see that our model structure is *finitely generated* and so [Hov99, Corollary 7.4.4] tells us that $\mathcal{D}(G) = \text{Ho}(\text{Ch}(\mathcal{G}))$ has a set of small weak generators. In other words, it is compactly generated. \square

Remark 1. Recall that by definition, a set \mathcal{S} of objects in a triangulated category such as $\mathcal{D}(G)$ is called a set of *weak generators* if $X = 0$ in $\mathcal{D}(G)$ if and only if $\mathcal{D}(G)(\Sigma^n S, X) = 0$ for all n and $S \in \mathcal{S}$. It is easy to see directly that $\{G_i = S^0(G_i)\}$ is a set of weak generators for $\mathcal{D}(G)$. Indeed we wish to see that X is G -acyclic if and only if $\mathcal{D}(G)(S^n(G_i), X) = 0$ for all n and i . But in the G -projective model structure we have that each $S^n(G_i)$ is cofibrant and every X is fibrant, so we get that $\mathcal{D}(G)(S^n(G_i), X) \cong \text{Ch}(\mathcal{G})(S^n(G_i), X)/\sim$ and the homotopy relation \sim is the usual relation of chain homotopic maps. So it all boils down to checking that X is G -acyclic if and only if $\text{Ch}(\mathcal{G})(S^n(G_i), X)/\sim = 0$ for all n and i . But this is clear upon noting that $\text{Ch}(\mathcal{G})(S^n(G_i), X)/\sim \cong H_n[\text{Hom}_{\mathcal{G}}(G_i, X)]$ and referring to Lemma 4.4.

4.5. Computation of $G\text{-Ext}_{\mathcal{G}}^n$. We have already seen an obvious analogy: G is to \mathcal{G}_G as R is to $R\text{-Mod}$. This analogy extends to the calculation of $G\text{-Ext}_{\mathcal{G}}^n(A, B)$, as the existence of the G -projective model structure formalizes the fact that one can do homological with respect to G . In more detail, according to Corollary 3.5, given any $A \in \mathcal{G}$, we may take a G -projective resolution

$$\mathcal{P} \twoheadrightarrow A \equiv \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \twoheadrightarrow A.$$

By this we mean it is G -acyclic and each P_n is G -projective. Then all the usual definitions and theorems hold for G -projective resolutions. For example, they are unique up to chain homotopy and one can define $G\text{-Ext}_{\mathcal{G}}^n(A, B)$ via such resolutions. We obtain long exact sequences, starting with G -exact sequences, etc. Moreover $G\text{-Ext}_{\mathcal{G}}^n(A, B)$ can alternately be defined using Yoneda's method: as equivalence classes of G -exact sequences $B \rightarrow L_1 \cdots \rightarrow L_n \twoheadrightarrow A$. (See also [CH02, Sections 1.2 and 2.1]; it is easy to see that the G -projectives and G -epimorphisms form a *projective class*.)

Our point here is that for a G -projective resolution $\mathcal{P} \twoheadrightarrow A$, we have a G -exact sequence of chain complexes $K \rightarrow \mathcal{P} \rightarrow S^0(A)$, where $K = \ker(\mathcal{P} \rightarrow S^0(A))$. Moreover K is G -acyclic and \mathcal{P} is semi- G -projective (since it is built up as a transfinite extension by consecutively attaching the semi- G -projective spheres $S^0(P_0)$, $S^1(P_1)$, $S^2(P_2)$, ...). So \mathcal{P} is a cofibrant replacement of $S^0(A)$ in the G -projective model structure. Hence using the fundamental theorem of model categories we have

$$\mathcal{D}(G)(A, \Sigma^n B) = \text{Ch}(\mathcal{G})(\mathcal{P}, S^n(B))/\sim = H^n[\text{Hom}_{\mathcal{G}}(\mathcal{P}, B)] = G\text{-Ext}_{\mathcal{G}}^n(A, B).$$

4.6. The λ -pure derived category. In [CH02, Section 5.3] we see the construction of a model structure for the pure derived category of a ring R and a canonical adjunction between the pure derived category of R and the usual derived category $\mathcal{D}(R)$. We describe now a natural extension of this fact to the G -derived category.

In any Grothendieck category \mathcal{G} , all objects are λ -presentable for some regular cardinal λ . In particular, for any choice of generator $G = \bigoplus G_i$ there is a λ such that all the G_i are λ -presentable. It follows that \mathcal{G} is locally λ -presentable. (See

Appendix A and [AR94, page 22] for language in this Subsection.) In fact, we see from [AR94, page 22] that \mathcal{G} is locally λ -presentable if and only if it has a generating set consisting of λ -presentable objects. Moreover, the category of all λ -presentable objects in \mathcal{G} has a small skeleton. So we can find a set $\{\Lambda_i\}$ of representatives from each isomorphism class, and we set $\Lambda = \oplus \Lambda_i$. Since $\{\Lambda_i\}$ contains $\{G_i\}$, it is also a generating set for \mathcal{G} . From Lemma 4.4 and Proposition A.1 the Λ -acyclic complexes are characterized as the exact complexes X for which each $Z_n X \subseteq X_n$ is λ -pure. Such complexes are called **λ -pure acyclic**. We call $\mathcal{D}(\Lambda)$ the **λ -pure derived category** of \mathcal{G} , and its model structure from Corollary 4.7 we call the **λ -pure projective model structure**. The extension groups of Subsection 4.5 we denote by $\lambda\text{-PExt}_{\mathcal{G}}^n$. We easily get the following.

Corollary 4.8. *Let \mathcal{G} be any Grothendieck category with G and Λ as above. There is a canonical functor $\mathcal{D}(\Lambda) \rightarrow \mathcal{D}(G)$ that is the identity on objects from the λ -pure derived category to the G -derived category. It induces a map $\lambda\text{-PExt}_{\mathcal{G}}^n(A, B) \rightarrow G\text{-Ext}_{\mathcal{G}}^n(A, B)$ which is natural in $A, B \in \mathcal{G}$. Moreover, $\mathcal{D}(\Lambda) \rightarrow \mathcal{D}(G)$ admits a left adjoint.*

Proof. First note that the identity functor $\text{Ch}(\mathcal{G}) \xrightarrow{\text{id}} \text{Ch}(\mathcal{G})$ is left adjoint to itself. Since $\{G_i\} \subseteq \{\Lambda_i\}$, the identity functor takes semi- G -projective complexes (complexes built from all the $S^n(G_i)$) to semi- Λ -projective complexes (complexes built from all the $S^n(\Lambda_i)$). Similarly it takes G -projective complexes (those built from the $D^n(G_i)$) to Λ -projective complexes (built from the $D^n(\Lambda_i)$). This directly leads us to conclude the identity functor is a left Quillen functor from the G -projective model structure to the λ -pure projective model structure. This automatically provides an adjunction $\mathcal{D}(G) \xrightarrow{L(\text{id})} \mathcal{D}(\Lambda)$, taking a complex X to its semi- G -projective cofibrant replacement. Its right adjoint $\mathcal{D}(\Lambda) \xrightarrow{R(\text{id})} \mathcal{D}(G)$ is the identity on objects since every object is fibrant. Since the functor $R(\text{id})$ is identity on objects, the functor provides, for all $A, B \in \mathcal{G}$, a natural map $\mathcal{D}(\Lambda)(A, \Sigma^n B) \rightarrow \mathcal{D}(G)(A, \Sigma^n B)$. But from Subsection 4.5 we see this translates to a natural map $\lambda\text{-PExt}_{\mathcal{G}}^n(A, B) \rightarrow G\text{-Ext}_{\mathcal{G}}^n(A, B)$. \square

5. THE INJECTIVE MODEL FOR LOCALLY FINITELY PRESENTABLE CATEGORIES

In the previous section, we constructed the G -derived category of any pair (\mathcal{G}, G) where \mathcal{G} is a Grothendieck category and $G = \oplus G_i$ is a generator. We constructed a model structure for $\mathcal{D}(G)$ in which the cofibrant complexes were built from G -projective objects. Our goal in this section is to construct a dual model structure for $\mathcal{D}(G)$, whose fibrant complexes are based on the G -injective objects. In order to do this we need to assume each G_i is finitely presented, or equivalently, that \mathcal{G} is **locally finitely presentable** (= locally ω -presentable as defined in Appendix A). Indeed from [AR94, Theorem 1.11] we have that \mathcal{G} is locally finitely presentable if and only if \mathcal{G} has a set of generators $\{G_i\}_{i \in I}$ for which each G_i is finitely presented (= ω -presented). Having different models for the same category is often useful. For example, the existence of the injective model structure implies the two recollement situations presented in Section 6.

5.1. G -homology in locally finitely presentable categories. For a chain complex X , we define its G -homology as $H_n[\text{Hom}_{\mathcal{G}}(G, X)]$. So the G -homology vanishes if and only if X is G -acyclic. Recall that in a general Grothendieck category,

a product of acyclic complexes need not again be acyclic. This is the point of Grothendieck's (AB4*) axiom. However, Theorem 4.6 tells us that the G -acyclic complexes are closed under products, since they are the right half of a cotorsion pair. The point of this subsection is to collect other useful properties that hold under the added assumption that each G_i is finitely presented. These properties will be used to construct the injective model structure on $\text{Ch}(\mathcal{G})$.

Lemma 5.1. *Assume each G_i is finitely presented. Up to a product, G -homology commutes with direct limits. That is, if $\{X_j\}_{j \in J}$ is a directed system of complexes, then*

$$H_n[\text{Hom}_{\mathcal{G}}(G, \varinjlim_{j \in J} X_j)] \cong \prod_{i \in I} \varinjlim_{j \in J} H_n[\text{Hom}_{\mathcal{G}}(G_i, X_j)]$$

If the set of generators $\{G_i\} = \{G_1, G_2, \dots, G_n\}$ is finite, then since direct limits commute with finite products we have

$$H_n[\text{Hom}_{\mathcal{G}}(G, \varinjlim_{j \in J} X_j)] \cong \varinjlim_{j \in J} H_n[\text{Hom}_{\mathcal{G}}(G, X_j)]$$

Proof. For complexes of abelian groups, homology commutes with products and direct limits. Also, the G_i are assumed finitely presented, so we have isomorphisms:

$$H_n[\text{Hom}_{\mathcal{G}}(G, \varinjlim_{j \in J} X_j)] \cong \prod_{i \in I} H_n[\text{Hom}_{\mathcal{G}}(G_i, \varinjlim_{j \in J} X_j)] \cong \prod_{i \in I} \varinjlim_{j \in J} H_n[\text{Hom}_{\mathcal{G}}(G_i, X_j)].$$

□

Proposition 5.2. *Assume each G_i is finitely presented. Then the following hold.*

- (1) *The G -acyclic complexes are closed under direct limits.*
- (2) *Direct limits of G -monomorphisms are again G -monomorphisms.*

Proof. The first statement follows from Lemma 5.1. For the second, suppose f is a monomorphism sitting in an exact sequence $\mathcal{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ which happens to be a directed limit of G -exact sequences $\mathcal{E}_j : 0 \rightarrow A_j \rightarrow B_j \rightarrow C_j \rightarrow 0$. Interpreting each \mathcal{E}_j as a G -acyclic chain complex the result follows from the first statement. □

By a *transfinite composition* we mean a map of the form $X_0 \xrightarrow{f} \varinjlim X_\alpha$ where $X : \lambda \rightarrow \mathcal{G}$ is a colimit-preserving functor and λ is an ordinal. In this case f is the transfinite composition of the $X_\alpha \rightarrow X_{\alpha+1}$. If each of these $X_\alpha \rightarrow X_{\alpha+1}$ is a G -monomorphism then f is a *transfinite composition of G -monomorphisms*. Furthermore, in this case we say that $\varinjlim X_\alpha$ is a *transfinite G -extension* of all the objects $X_0, X_{\alpha+1}/X_\alpha$.

Corollary 5.3. *Assume each G_i is finitely presented. Then the following hold.*

- (1) *The G -acyclic complexes are closed under transfinite G -extensions and direct sums.*
- (2) *An arbitrary transfinite composition of G -monomorphisms is again a G -monomorphism.*

Proof. The G -acyclic complexes are always closed under G -extensions by Lemma 4.4. So they are closed under transfinite G -extensions by Proposition 5.2. Direct sums are special cases of transfinite G -extensions.

For the second statement, we first note that a finite composition ($\lambda = n \in \mathbb{N}$) of G -monomorphisms is again a G -monomorphism by part (4) of Proposition 3.3. For

$\lambda = \omega$, we want the map $X_0 \xrightarrow{f_\omega} \varinjlim_{n < \omega} X_n$ to also be a G -monomorphism. So we want the short exact sequence

$$\mathcal{E} : 0 \rightarrow X_0 \xrightarrow{f_\omega} \varinjlim_{n < \omega} X_n \rightarrow (\varinjlim_{n < \omega} X_n)/X_0 \rightarrow 0$$

to be G -exact. But this is the direct limit of the short exact sequences

$$\mathcal{E}_n : 0 \rightarrow X_0 \xrightarrow{f_n} X_n \rightarrow X_n/X_0 \rightarrow 0$$

and these are G -exact because this is the finite case $\lambda = n$. So the $\lambda = \omega$ case holds by Proposition 5.2. We see the result follows by transfinite induction. \square

5.2. Complete cotorsion pairs. The result here is taken, with only a few small adjustments for our situation, from the original source [Hov02]. We again use the notion of a *small* cotorsion pair from [Hov02] as well as the notation I -cell and I -inj from [Hov99].

Proposition 5.4. *Consider the G -exact category \mathcal{G}_G in the case that each G_i is finitely presented. Then a cotorsion pair $(\mathcal{F}, \mathcal{C})$ in \mathcal{G}_G is cogenerated by a set \mathcal{S} if and only if it is small with generating monomorphisms the set*

$$I = \{0 \rightarrow G_i\}_{i \in I} \cup \{K_S \rightarrow P_S \rightarrow S\}_{S \in \mathcal{S}}.$$

Here we have chosen for each $S \in \mathcal{S}$, a G -exact sequence $K_S \rightarrow P_S \rightarrow S$ with P_S a G -projective object. Such a cotorsion pair $(\mathcal{F}, \mathcal{C})$ satisfies each of the following:

- (1) $(\mathcal{F}, \mathcal{C})$ is functorially complete.
- (2) \mathcal{F} consists precisely of retracts of transfinite G -extensions of \mathcal{S} .
- (3) I -inj is precisely the class of all G -epimorphisms with kernel in \mathcal{C} .

Proof. Note that we can find the G -exact sequences $K_S \rightarrow P_S \rightarrow S$ with each P_S a G -projective by using Corollary 3.5. We see that the functors $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1(P_S, -)$ and $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1(G_i, -)$ vanish. So it is easy to see that \mathcal{S} cogenerates the cotorsion pair iff the given set I forms a set of generating monomorphisms in the sense of [Hov02, Definition 6.4].

By Corollary 5.3 we have that transfinite compositions of G -monomorphisms are again G -monomorphisms. So by [Hov02, Theorem 6.5] we get that $(\mathcal{F}, \mathcal{C})$ is a functorially complete cotorsion pair. The proof there shows that \mathcal{F} consists precisely of retracts of transfinite G -extensions of objects in \mathcal{S} .

It is left to see that I -inj is precisely the class of all G -epimorphisms with kernel in \mathcal{C} . Showing that everything in I -inj is a G -epimorphism with kernel in \mathcal{C} is formally similar to the first claim in the proof of Theorem 4.6. The converse is similar to the argument given in the last paragraph of the proof of Corollary 4.7. We leave the details to the reader. \square

Remark 2. We note that Proposition 5.4 applies not just to \mathcal{G}_G but also to $\text{Ch}(\mathcal{G})_G$ by Lemma 4.1. This is because each $D^n(G_i)$ is a finitely presented complex whenever each G_i is finitely presented. In particular, any cotorsion pair in $\text{Ch}(\mathcal{G})_G$ that is cogenerated by a set is complete.

5.3. Injectives in \mathcal{G}_G and $\text{Ch}(\mathcal{G})_G$. We need to show that the exact categories \mathcal{G}_G and $\text{Ch}(\mathcal{G})_G$ have enough injective objects. Following our language for the projective case, we will call these objects *G-injective*. We will use the theory of purity summarized in Appendix A. The appendix shows that when \mathcal{G} is locally finitely presentable (= locally ω -presentable) we have a well-behaved notion of **pure** (= ω -pure) subobjects $P \subseteq X$ in \mathcal{G} . In particular, we get that pure monomorphisms are closed under directed colimits (= ω -directed colimits) in \mathcal{G} by Proposition A.1.

Note that any pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{G} is automatically a G -exact sequence. This follows from Proposition A.1, our assumption that each G_i is finitely presented, and the fact that direct products of short exact sequences (of abelian groups) are still short exact sequences. In particular, any pure subobject is automatically a G -subobject.

Remark 3. For *any* Grothendieck category \mathcal{G} there exist arbitrarily large regular cardinals λ such that the λ -presented objects coincide with the λ -generated objects. The author thanks Jiří Rosický for providing the following reason for this statement: Let $\mathcal{G}_{\text{mono}}$ denote the category consisting of the same objects as \mathcal{G} but with morphisms only the monomorphisms of \mathcal{G} . Then for any λ , the λ -presented objects of $\mathcal{G}_{\text{mono}}$ coincide exactly with the λ -generated objects of \mathcal{G} . Moreover we note $\mathcal{G}_{\text{mono}}$ is an accessible category by [AR94, Local Generation Theorem 1.70]. The embedding functor $\mathcal{G}_{\text{mono}} \rightarrow \mathcal{G}$ is an *accessible functor* in the sense of [AR94, Definition 2.16]. Therefore, the Uniformization Theorem [AR94, Theorem 2.19 and Remark] applies which means there are arbitrarily large regular cardinals λ for which this embedding is λ -accessible and preserves λ -presented objects. This means exactly that there exist arbitrarily large regular cardinals λ such that the λ -presented objects coincide with the λ -generated objects. In fact, it follows from [AR94, Remark 2.20] that if γ is a regular cardinal for which $\lambda \triangleleft \gamma$, that is λ is *sharply smaller* than γ in the sense of [AR94, Definition 2.12], then the γ -presented objects coincide with the γ -generated objects too.

Note that for any γ as in Remark 3 the notion of γ -presented (= γ -generated) becomes a substitute for “cardinality $< \gamma$ ”. In particular, the class of γ -presented objects is closed under quotients and subobjects. We also have that, up to isomorphism, there is just a set of γ -presented objects.

Setup 5.5. We now specify for our locally finitely presentable category \mathcal{G} a regular cardinal γ which will be of use. We fix a regular cardinal γ with each of the following properties:

- (1) The γ -presented objects coincide with the γ -generated objects.
- (2) Whenever we have a subobject $S \subseteq X$ where S is γ -generated, there exists a pure subobject $P \subseteq X$ which is also γ -generated and which contains S .

Lets now justify why we can choose such a cardinal γ . First, from the above Remark 3 we can find a regular cardinal λ such that whenever γ is a regular cardinal with $\lambda \triangleleft \gamma$, then the γ -presented objects coincide with the γ -generated objects. Since our category \mathcal{G} is locally ω -presentable it is also locally λ -presentable, (since $\omega \leq \lambda$ and [AR94, Remark 1.20]). So by [AR94, Theorem 2.33] we are guaranteed the existence of arbitrarily large regular cardinals $\gamma \triangleright \lambda$ with the following property: Whenever we have a subobject $S \subseteq X$ where S is γ -generated, there exists a λ -pure subobject $P \subseteq X$ which is also γ -generated and which contains S . However, any λ -pure P is also pure because $\omega \leq \lambda$ and [AR94, Remark (3) pp. 85]. (However,

we warn the reader that there is a misprint in [AR94, Remark (3) pp. 85]. The inequality goes the other way.) But we are done.

The main purpose for constructing γ in Setup 5.5 is to use its properties (i) and (ii) to show that any G -acyclic complex is a transfinite G -extension of γ -presented G -acyclic complexes. Although perhaps overkill, we will also now use γ to show that \mathcal{G}_G has enough injectives.

Proposition 5.6. *Let γ be as in Setup 5.5 and let \mathcal{S} be a set of isomorphic representatives for the class of all γ -presented objects. Then \mathcal{S} cogenerates the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ in \mathcal{G}_G . That is, \mathcal{A} consists of all objects of \mathcal{G} , while $\mathcal{I} = \mathcal{S}^\perp$ is precisely the class of injective objects of \mathcal{G}_G . We call these objects **G -injective**. $(\mathcal{A}, \mathcal{I})$ is complete, meaning \mathcal{G}_G has enough G -injectives.*

Proof. By Proposition 5.4 we know that \mathcal{S} cogenerates a complete cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ where ${}^\perp(\mathcal{S}^\perp)$ consist precisely of retracts of transfinite G -extensions of \mathcal{S} . Letting \mathcal{A} denote the class of all objects of \mathcal{G} we will be done if we can show $\mathcal{A} \subseteq {}^\perp(\mathcal{S}^\perp)$. By [Hov02, Lemma 6.2] it suffices to show that every object in \mathcal{A} is a transfinite G -extension of objects in \mathcal{S} . But since each G_i is finitely presented, we note that pure exact sequences are automatically G -exact. So it is enough to show that any object is a transfinite pure-extension of γ -presented objects.

So let M be any given object. First note that assuming $M \neq 0$, we can always find a nonzero pure subobject $P_0 \subseteq M$ with P_0 γ -presented. Assuming $P_0 \neq M$, we can do the same to M/P_0 to get a pure $P_1/P_0 \subseteq M/P_0$ with P_1/P_0 γ -presented. Assuming we are not done, we continue to construct a strictly increasing $0 \neq P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots$. Note each $P_n \subseteq M$ is pure by part (3) of Proposition A.2. Then set $P_\omega = \bigcup_{n < \omega} P_n$ and note it is also pure by part (4) of Proposition A.2. In this way we continue by transfinite induction to get $M = \bigcup_{\alpha < \lambda} P_\alpha$ for some λ where each $P_\alpha \subseteq P_{\alpha+1}$ is pure. \square

Remark 4. No matter what our choice is for the generator $G = \bigoplus_{i \in I} G_i$, it is the same set \mathcal{S} cogenerating the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ (as long as each G_i is finitely presented). But a different choice of generating set $\{G_i\}$ will of course change the proper class of short exact sequences in \mathcal{G}_G . Consequently, this changes the class \mathcal{S}^\perp of G -injectives. (It of course also changes the G -projectives).

Note that because of Lemma 4.1, the above Proposition 5.6 also applies to the chain complex category $\text{Ch}(\mathcal{G})_G$. That is, there are enough G -injective complexes. As in Lemma 4.5 we have the following classification of G -injective complexes.

Lemma 5.7. *Call a chain complex X in $\text{Ch}(\mathcal{G})$ a **G -injective** complex if it is injective in the exact category $\text{Ch}(\mathcal{G})_G$. The following are equivalent:*

- (1) X is G -injective.
- (2) X is G -acyclic with each $Z_n X$ a G -injective.
- (3) X is isomorphic to a split exact complex with G -injective components. That is, $X \cong \bigoplus_{n \in \mathbb{Z}} D^n(I_n)$ where each I_n is a G -injective.
- (4) X is a contractible complex with each X_n G -injective.

We note that there are enough G -injective complexes. This follows from Proposition 5.6 and Lemma 4.1.

5.4. The injective model structure. We now wish to construct an injective model structure for the G -derived category, assuming each G_i is finitely presented.

The following lemma, which holds for arbitrary Grothendieck categories, will be used in the main proof. It is a generalization of [Sten75, Lemma V.3.3].

Lemma 5.8. *Let \mathcal{G} be a locally λ -presentable Grothendieck category. Given an epimorphism $g: X \rightarrow Y$ where Y is λ -generated, there exists a λ -generated subobject $X' \subseteq X$ for which $g|_{X'}: X' \rightarrow Y$ is an epimorphism.*

Proof. Any locally λ -presentable Grothendieck category is also locally λ -generated. This means that, up to isomorphism, there is a set of λ -generated objects and that every object is a λ -directed union of its λ -generated subobjects. (The proof of this goes by writing the given object $X = \varinjlim X_i$ as a λ -directed colimit of λ -presented X_i . Then factor each $X_i \rightarrow X$ as an epi followed by a mono. Each $\text{Im } X_i$ is λ -generated and X_i is the λ -directed union of the $\text{Im } X_i$.) So we may write $X = \sum_{i \in I} X_i$ as a λ -directed union of λ -generated subobjects of X . Since g is an epimorphism, $Y = \sum_{i \in I} g(X_i)$, and this too is a λ -directed union. Now we must have $Y = g(X_i)$ for some $i \in I$ since Y is λ -generated. So $g|_{X_i}: X_i \rightarrow Y$ is an epimorphism. \square

Recall (see Lemma 4.4), that a chain complex X is G -acyclic if and only if it is exact and each $Z_n X$ is a G -subobject of X_n . This means the inclusion $Z_n X \hookrightarrow X_n$ is a G -monomorphism, and we write $Z_n X \subseteq_G X_n$.

Lemma 5.9. *Let γ be as in Setup 5.5. Given any nonzero G -acyclic complex E there exists a degreewise G -exact sequence $P \hookrightarrow E \twoheadrightarrow E/P$ where P is a nonzero G -acyclic subcomplex with each P_n γ -presented.*

Proof. (Step 1) We first prove the following: For any given n and exact $S \subseteq E$ with each S_i γ -presented, there exists an exact $T \subseteq E$ satisfying the following:

- (1) $S \subseteq T$ and each T_i is γ -presented.
- (2) $Z_n T \subseteq_G T_n$ is a G -subobject.
- (3) $S_n \subseteq P \subseteq T_n \subseteq E_n$ for some G -subobject $P \subseteq E_n$.

Indeed as in Setup 5.5 we can find a γ -presented pure $P \subseteq E_n$ containing S_n . Then set $T_{n-1} = S_{n-1} + d(P)$ and note that it is γ -presented and that $\ker d|_{T_{n-1}} = d(P)$. We set $T_{n-2} = S_{n-2}$, $T_{n-3} = S_{n-3}$, etc. going downward. This gives us a portion of a subcomplex we are building

$$\cdots P \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots$$

which we note is exact in degrees $n-1$ and below. We wish to extend upwards to an exact complex.

Note that $\ker d|_P$ is also γ -presented. So there exists a γ -presented pure subobject $P' \subseteq Z_n E$ containing $\ker d|_P$. Now let $T_n = P + P'$, and note that we still have exactness in degrees $\leq n-1$ in the (still unfinished) subcomplex shown

$$\cdots T_n \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots$$

Moreover, since $\ker d|_{T_n} = P'$ is pure in $Z_n E$, it is a G -subobject $\ker d|_{T_n} = P' \subseteq_G Z_n E$. We also have $Z_n E \subseteq_G E_n$ by assumption, and so from part (1) of Proposition 3.6 we have $\ker d|_{T_n} = P' \subseteq_G E_n$. But then from part (2) of Proposition 3.6 we have $\ker d|_{T_n} = P' \subseteq_G T_n$. (Here we have arranged conditions (2) and (3) in the subcomplex T that we are constructing.)

Now since P' is γ -presented, we can use Lemma 5.8 to find a γ -presented subobject $S'_{n+1} \subseteq E_{n+1}$ for which $d|_{S'_{n+1}}: S'_{n+1} \rightarrow P'$ is an epimorphism. We set

$T_{n+1} = S_{n+1} + S'_{n+1}$ and note that

$$\cdots T_{n+1} \rightarrow T_n \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots$$

is now exact in degrees n and below. Repeatedly using Lemma 5.8 in this way we can continue upward to obtain an exact subcomplex $T \subseteq E$ which contains S , which has each T_i γ -presented, has $Z_n T = P' \subseteq_G T_n$, and has $S_n \subseteq P \subseteq T_n \subseteq E_n$ where $P \subseteq_G E_n$.

(Step 2) We now complete the proof. For the construction just described in (Step 1), let's say that the complex T was obtained by applying a “degree n operation to S ”. Start by first finding *any* nonzero exact complex $S \subseteq E$ with each S_i γ -presented, and with this S apply a “degree 0 operation to S ” to obtain a T^0 with $S \subseteq T^0 \subseteq E$ and the guaranteed properties in (Step 1). Then apply a “degree -1 operation to T^0 ” to obtain a complex T^1 with $T^0 \subseteq T^1 \subseteq E$. Then again apply a “degree 0 operation to T^1 to obtain a T^2 . We continue to use “degree k operations” on the previously constructed complex in the following back and forth pattern on k :

$$0, \quad -1, 0, 1, \quad -2, -1, 0, 1, 2, \quad -3, -2, -1, 0, 1, 2, 3, \quad \cdots$$

to build an increasing union of exact subcomplexes, $\{T^l\}$. Finally, set $P = \cup_{l \in \mathbb{N}} T^l$. We now verify that P has the desired properties:

- (1) $S \subseteq P \subseteq E$ and each P_n is γ -presented.
(Reason) The containments are clear and each $P_n = \cup_{l \in \mathbb{N}} (T^l)_n$ is γ -presented because all the $(T^l)_n$ are γ -presented and $|\mathbb{N}| < \gamma$. See [AR94, Proposition 1.16].
- (2) P is G -acyclic.
(Reason) P is exact since it is a direct union of exact subcomplexes. Moreover each $Z_n P = \cup_{l \in \mathbb{N}} Z_n(T^l)$ must be a G -subobject of P_n by Proposition 5.2 as the union contains a cofinal collection of G -monomorphisms.
- (3) $P_n \subseteq_G E_n$ for each n .
(Reason) Each $P_n = \cup_{l \in \mathbb{N}} (T^l)_n$ must be a G -subobject of E_n because again, this union contains a cofinal collection of G -subobjects of E_n by property (3) in (Step 1).

□

Proposition 5.10. *Let γ be as in Setup 5.5. Each G -acyclic complex is a transfinite G -extension of γ -presented G -acyclic complexes.*

Proof. Suppose $E \neq 0$ is G -acyclic and use Lemma 5.9 to find a nonzero γ -presented G -acyclic subcomplex $0 \neq P_0 \subseteq E$ which is a G -subobject in each degree. Then applying $\text{Hom}_{\mathcal{G}}(G, -)$ to $P_0 \hookrightarrow E \twoheadrightarrow E/P_0$ leaves an exact sequence of complexes and it follows that E/P_0 is G -acyclic also. Assuming this complex is not zero find another nonzero γ -presented G -acyclic complex $P_1/P_0 \subseteq E/P_0$ which is a G -subobject in each degree. Since $P_0 \subseteq E$ is a G -subobject in each degree, we get that $P_0 \subseteq P_1$ is also a G -subobject in each degree by Proposition 3.6, part (2). Then part (3) of that same Proposition tells us that $P_1 \subseteq E$ is a G -subobject in each degree. Assuming $P_1 \neq E$, we continue to find an increasing sequence $0 \neq P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots$ of G -acyclic subcomplexes of E with each $P_n \subseteq E$ a G -subobject in each degree. Then set $P_\omega = \cup_{n < \omega} P_n$ and we see that P_ω is too a G -acyclic complex by Proposition 5.2 and also $P_\omega \subseteq E$ is a G -subobject in each degree, again by Proposition 5.2. Therefore E/P_ω is also G -acyclic and we can

continue with transfinite induction to end up with E displayed as a transfinite G -extension of γ -presented G -acyclic complexes. \square

Theorem 5.11. *Let \mathcal{G} be a Grothendieck category with a generator $G = \bigoplus_{i \in I} G_i$ where each G_i is finitely presented. Let \mathcal{W} be the class of all G -acyclic complexes. Then there is an injective cotorsion pair $(\mathcal{W}, \mathcal{I})$ in $\text{Ch}(\mathcal{G})_G$. That is, it is a complete cotorsion pair in $\text{Ch}(\mathcal{G})_G$ for which \mathcal{W} is thick in $\text{Ch}(\mathcal{G})_G$ and $\mathcal{W} \cap \mathcal{I}$ coincides with the class of injective complexes in $\text{Ch}(\mathcal{G})_G$. We call the complexes in \mathcal{I} the **semi- G -injective** complexes.*

Proof. Let γ be as in Setup 5.5 and take \mathcal{S} to be a set of isomorphism representatives for the class of all γ -presented complexes in \mathcal{W} . So everything in \mathcal{S} is a G -acyclic complex S with each S_n being γ -presented. We will show that \mathcal{S} cogenerates $(\mathcal{W}, \mathcal{I})$ in $\text{Ch}(\mathcal{G})_G$. Recall that cotorsion pairs in $\text{Ch}(\mathcal{G})_G$ are with respect to $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1$. By Remark 2 which follows Proposition 5.4, we know that \mathcal{S} cogenerates a complete cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ in $\text{Ch}(\mathcal{G})_G$ where ${}^\perp(\mathcal{S}^\perp)$ consists precisely of retracts of transfinite G -extensions of \mathcal{S} . We wish to show $\mathcal{W} = {}^\perp(\mathcal{S}^\perp)$. But we already know that \mathcal{W} is thick in $\text{Ch}(\mathcal{G})_G$ by Lemma 4.4 and closed under transfinite G -extensions by Corollary 5.3. So $\mathcal{W} \supseteq {}^\perp(\mathcal{S}^\perp)$. On the other hand, $\mathcal{W} \subseteq {}^\perp(\mathcal{S}^\perp)$ was proved in Proposition 5.10. So $(\mathcal{W}, \mathcal{I})$ is a complete cotorsion pair where $\mathcal{I} = \mathcal{S}^\perp$.

Since we already know \mathcal{W} is thick, all that is left is to show that $\mathcal{W} \cap \mathcal{I}$ coincides with the class of injective complexes in $\text{Ch}(\mathcal{G})_G$. But by the argument in [BGH13, Proposition 3.3] it is enough to show that the injectives in $\text{Ch}(\mathcal{G})_G$ are contained in \mathcal{W} . Since the injective complexes are precisely the contractible complexes with G -injective components by Lemma 5.7, these are in \mathcal{W} by lemma 4.4. \square

The following corollary now follows from the main result in [Hov02].

Corollary 5.12. *Let \mathcal{G} be a Grothendieck category with a generator $G = \bigoplus_{i \in I} G_i$ where each G_i is finitely presented. Then there is a model structure on $\text{Ch}(\mathcal{G})$ which we call the **G -injective model structure** whose trivial objects are the G -acyclic complexes. The model structure satisfies the following:*

- (1) *The cofibrations are precisely the G -monomorphisms. That is, the chain maps which are G -monomorphisms in each degree.*
- (2) *The trivial cofibrations are the G -monomorphisms with G -acyclic cokernel.*
- (3) *The fibrations are the degreewise split epimorphisms whose kernel is a semi- G -injective complex.*
- (4) *The trivial fibrations are the split epimorphisms whose kernel is a G -injective complex.*
- (5) *The weak equivalences are the G -homology isomorphisms.*
- (6) *The model structure is cofibrantly generated. Sets of generating cofibrations and generating trivial cofibrations can be found using Proposition 5.4.*
- (7) *The homotopy category is equivalent to $\mathcal{D}(G)$, and this is a compactly generated triangulated category by Corollary 4.7.*

6. THE RECOLLEMENT SITUATIONS

Again, \mathcal{G} is a locally finitely presentable Grothendieck category with generator $G = \bigoplus_{i \in I} G_i$ where each G_i is finitely presented. Here we wish to prove the two recollement situations from Theorems B and C of the Introduction.

We will use the correspondence between injective (resp. projective) cotorsion pairs and recollements situations from [Gil12] and [Gil13]. By definition, a cotorsion pair $(\mathcal{P}, \mathcal{W})$ in \mathcal{G}_G (or $\text{Ch}(\mathcal{G})_G$) is a **projective cotorsion pair** if it is complete, \mathcal{W} is G -thick, and if $\mathcal{P} \cap \mathcal{W}$ coincides with the class of G -projective objects. Since the category \mathcal{G}_G has enough projectives this makes the triple $(\mathcal{P}, \mathcal{W}, \mathcal{A})$, where \mathcal{A} represents the class of all objects, correspond to a model structure on \mathcal{G} via Hovey's correspondence [Hov02, Theorem 2.2]. For example, the cotorsion pair of Theorem 4.6 is a projective cotorsion pair in $\text{Ch}(\mathcal{G})_G$ and corresponds to the model structure of Corollary 4.7. On the other hand, we showed in Proposition 5.6 that \mathcal{G}_G (and so $\text{Ch}(\mathcal{G})_G$) also has enough injectives and so it also makes sense to speak of **injective cotorsion pairs** which are the dual. For example, the cotorsion pair of Theorem 5.11 is an injective cotorsion pair in $\text{Ch}(\mathcal{G})_G$ and gave us the model structure of Corollary 5.12.

Proposition 6.1. *Assume each G_i is finitely presented. There is an injective model structure $(\mathcal{W}_1, \mathcal{F}_1)$ in $\text{Ch}(\mathcal{G})_G$ where \mathcal{F}_1 is the class of all complexes of G -injective complexes.*

Proof. From Proposition 5.4 and Remark 2 which follows it, we know that *any* set of complexes cogenerates a complete cotorsion pair in $\text{Ch}(\mathcal{G})_G$. Here we let $\mathcal{S}_1 = \{D^n(S) \mid S \in \mathcal{S}\}$ where \mathcal{S} is the set in Proposition 5.6 which cogenerates the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ in \mathcal{G}_G . So \mathcal{I} is the class of G -injectives. By Lemma 4.2 we have $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(D^n(S), X) \cong \text{G-Ext}_{\mathcal{G}}^1(S, X_n)$. It follows that $\mathcal{S}_1^\perp = \mathcal{F}_1$ in $\text{Ch}(\mathcal{G})_G$. So we get a complete cotorsion pair $(\mathcal{W}_1, \mathcal{F}_1)$ in $\text{Ch}(\mathcal{G})_G$ where \mathcal{F}_1 is the class of all complexes of G -injective complexes.

To show it is an injective cotorsion pair in $\text{Ch}(\mathcal{G})_G$, we only need to show that \mathcal{W}_1 is G -thick and contains the injectives. Note that for any complex W and $F \in \mathcal{F}_1$ we have $\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(W, F) = \text{Ext}_{dw}^1(W, F)$. So by Lemma 4.3, $W \in \mathcal{W}_1$ if and only if $\text{Hom}(W, F)$ is exact. So to see that \mathcal{W}_1 is G -thick we consider a degreewise G -exact sequence of complexes $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$. Then as noted earlier, for any complex F of G -injectives, applying $\text{Hom}(-, F)$ will give us a short exact sequence $0 \rightarrow \text{Hom}(Z, F) \rightarrow \text{Hom}(Y, F) \rightarrow \text{Hom}(X, F) \rightarrow 0$. So if two out of the three complexes are exact, then so is the third. This proves thickness of \mathcal{W}_1 in $\text{Ch}(\mathcal{G})_G$. If I is an injective complex in $\text{Ch}(\mathcal{G})_G$, then by Lemma 5.7 it is a split exact complex with G -injective components. In particular, it is contractible. So for such an I we have $\text{Hom}(I, F)$ is exact for any $F \in \mathcal{F}_1$. \square

Proposition 6.2. *Assume each G_i is finitely presented. There is an injective model structure $(\mathcal{W}_2, \mathcal{F}_2)$ in $\text{Ch}(\mathcal{G})_G$ where \mathcal{F}_2 is the class of all G -acyclic complexes of G -injectives.*

Proof. Take \mathcal{S}_1 from the proof of Proposition 6.1 and let $\mathcal{S}_2 = \mathcal{S}_1 \cup \{S^n(G)\}$. We claim that $\mathcal{S}_2^\perp = \mathcal{F}_2$ in $\text{Ch}(\mathcal{G})_G$. Indeed if $X \in \mathcal{S}_2^\perp$ then X is a complex of G -injectives for which $0 = \text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(S^n(G), X) = \text{Ext}_{dw}^1(S^n(G), X) = H_{n-1}\text{Hom}(S^0(G), X) = H_{n-1}\text{Hom}_{\mathcal{G}}(G, X)$. So X is G -acyclic. Conversely, if X is G -acyclic with G -injective components then $X \in \mathcal{S}_2^\perp$. So we get a complete cotorsion pair by again applying Proposition 5.4 and Remark 2 which follows it. The fact that \mathcal{W}_2 is thick and contains the G -injective complexes follows just like in Proposition 6.1. \square

Theorem 6.3 (Krause’s recollement for G -derived categories). *Assume each G_i is finitely presented. Let $\mathcal{D}(G)$ denote the G -derived category. Let $K_G(\text{Inj})$ denote the homotopy category of all complexes of G -injectives. Let $K_{G\text{-ac}}(\text{Inj})$ denote the homotopy category of all G -acyclic complexes of G -injectives. Then there is a recollement*

$$K_{G\text{-ac}}(\text{Inj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} K_G(\text{Inj}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}(G).$$

Proof. Take $(\mathcal{W}_1, \mathcal{F}_1)$ to be the injective cotorsion pair from Proposition 6.1. Take $(\mathcal{W}_2, \mathcal{F}_2)$ to be the injective cotorsion pair from Proposition 6.2. Take $(\mathcal{W}_3, \mathcal{F}_3) = (\mathcal{W}, \mathcal{I})$ to be the semi- G -injective cotorsion pair from Theorem 5.11. Since $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ the result is automatic from [Gil13, Theorem 3.4]. \square

Theorem 6.4 (Verdier localization recollement for G -derived categories). *Assume each G_i is finitely presented. Let $\mathcal{D}(G)$ denote the G -derived category. Let $K(\mathcal{G})$ denote the homotopy category of all chain complexes and let $K_{G\text{-ac}}(\mathcal{G})$ denote the subcategory of all G -acyclic complexes. Then there is a recollement*

$$\begin{array}{ccccc} & E(K\mathcal{P}, \mathcal{W}) & & \lambda = C(K\mathcal{P}, \mathcal{W}) & \\ & \swarrow & & \swarrow & \\ K_{G\text{-ac}}(\mathcal{G}) & \xrightarrow{I} & K(\mathcal{G}) & \xrightarrow{Q} & \mathcal{D}(G) \\ & \searrow & & \searrow & \\ & C(\mathcal{W}, K\mathcal{I}) & & \rho = E(\mathcal{W}, K\mathcal{I}) & \end{array}$$

Here, \mathcal{W} is the class of G -acyclic complexes, and the complexes in $K\mathcal{P}$ are the G -analog of Spaltenstein’s K -projective complexes. The functor $C(K\mathcal{P}, \mathcal{W})$ is the functor taking X to its $K\mathcal{P}$ -precover since $(K\mathcal{P}, \mathcal{W})$ turns out to be a complete cotorsion pair in $\text{Ch}(\mathcal{G})_{dw}$. Similarly $K\mathcal{I}$ is analogous to the class of K -injective complexes and $E(\mathcal{W}, K\mathcal{I})$ is the functor taking X to its $K\mathcal{I}$ -preenvelope.

Proof. The basic idea is that the existence of the G -projective model $(\mathcal{P}, \mathcal{W})$ of Section 4 provides a left adjoint to the inclusion $K_{G\text{-ac}}(\mathcal{G}) \rightarrow K(\mathcal{G})$, and in fact a colocalization sequence $K_{G\text{-ac}}(\mathcal{G}) \rightarrow K(\mathcal{G}) \rightarrow \mathcal{D}(G)$. On the other hand, the existence of the G -injective model $(\mathcal{W}, \mathcal{I})$ of Section 5 provides a right adjoint to the inclusion $K_{G\text{-ac}}(\mathcal{G}) \rightarrow K(\mathcal{G})$, and in fact a localization sequence $K_{G\text{-ac}}(\mathcal{G}) \rightarrow K(\mathcal{G}) \rightarrow \mathcal{D}(G)$. Together this is a recollement. The formalization in terms of model structures follows immediately from work in [Gil13, Section 6]. The theory there is all written in terms of weakly idempotent complete exact categories, and so applies to our current setting. In full detail, we apply [Gil13, Theorem 6.3] to the G -injective model structure $(\mathcal{W}, \mathcal{I})$ to obtain a Quillen equivalent model structure $(\mathcal{W}, K\mathcal{I})$ in the exact category $\text{Ch}(\mathcal{G})_{dw}$ of chain complexes with degreewise split short exact sequences. The complexes in $K\mathcal{I}$ are the G -analog of Spaltenstein’s K -injective complexes and in fact are, by [Gil13, Proposition 6.4], precisely the complexes that are chain homotopy equivalent to a semi- G -injective complex. The dual of [Gil13, Theorem 6.3] applied to the G -projective model structure $(\mathcal{P}, \mathcal{W})$ gives us a similar model $(K\mathcal{P}, \mathcal{W})$. All together $(K\mathcal{P}, \mathcal{W}, K\mathcal{I})$ is *localizing cotorsion triple* in the sense of [Gil13, Section 4.1] and so by [Gil13, Corollary 4.5] we obtain the recollement. \square

APPENDIX A. λ -PURITY IN GROTHENDIECK CATEGORIES

Every Grothendieck category \mathcal{G} is locally presentable. This means there exists a regular cardinal λ and a set \mathcal{S} of λ -presented objects such that every object of \mathcal{G} is a λ -directed colimit of objects of \mathcal{S} . In this case we say \mathcal{G} is locally λ -presentable and it is true that for any regular cardinal $\lambda' > \lambda$, we have \mathcal{G} is locally λ' -presentable as well. See [AR94, Theorem 1.20 and the Remark].

Now following [AR94], a morphism f is called λ -pure if for each commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ u \downarrow & & \downarrow v \\ A & \xrightarrow{f} & B \end{array}$$

with A', B' λ -presented there is a map t such that $u = tf'$. Assuming the category is locally λ -presentable we have from [AR94, Proposition 2.29] that a λ -pure morphism must be a monomorphism. In fact, they are characterized in [AR94, Proposition 2.30 and its Corollary] as being precisely the λ -directed colimits (in the category of morphisms) of split monomorphisms. Since Grothendieck categories are abelian we are lead naturally to speak instead of λ -pure short exact sequences, which we now characterize.

Proposition A.1 (λ -purity in Grothendieck categories). *Let \mathcal{G} be a locally λ -presentable Grothendieck category and let $\mathcal{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence. Then the following are equivalent and characterize what we mean by saying \mathcal{E} is a λ -pure short exact sequence.*

- (1) f is a λ -pure morphism.
- (2) $\mathrm{Hom}_{\mathcal{G}}(L, \mathcal{E})$ is a short exact sequence of abelian groups for any λ -presented object L .
- (3) \mathcal{E} is a λ -directed limit of split short exact sequences

$$\mathcal{E}_i : 0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0 \quad (i \in I).$$

Proof. As already pointed out above, we have from [AR94, Proposition 2.30 and Corollary] that the λ -pure morphisms are precisely the λ -directed colimits of split monomorphisms. In particular, if f is a λ -pure morphism, we get that the short exact sequence

$$\mathcal{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

must be a λ -directed colimit of split short exact sequences

$$\mathcal{E}_i : 0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0.$$

So (1) if and only if (3). But if (3) holds, then we clearly have that each $\mathrm{Hom}_{\mathcal{G}}(L, \mathcal{E}_i)$ is exact for any L . If L is λ -presented then $\mathrm{Hom}_{\mathcal{G}}(L, \mathcal{E}) \cong \varinjlim \mathrm{Hom}_{\mathcal{G}}(L, \mathcal{E}_i)$ is also exact. So (3) implies (2).

Now we show (2) implies (3). Using that \mathcal{G} is locally λ -presentable, write $C = \varinjlim C_i$ as a λ -directed colimit of λ -presented C_i . For each $\gamma_i : C_i \rightarrow C$, form the pullback

$$\begin{array}{ccccccc} \mathcal{E}_i : & 0 & \longrightarrow & A & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow \gamma_i & & \\ \mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

If (2) holds, then γ_i lifts over g . This implies that \mathcal{E}_i splits. One can check that $\mathcal{E} \cong \varinjlim \mathcal{E}_i$. \square

Proposition A.2. *Let \mathcal{G} be a locally λ -presentable Grothendieck category and $A \subseteq B \subseteq C$.*

- (1) *If $A \subseteq B$ is λ -pure and $B \subseteq C$ is λ -pure then $A \subseteq C$ is λ -pure.*
- (2) *If $A \subseteq C$ is λ -pure then $A \subseteq B$ is λ -pure.*
- (3) *If $A \subseteq C$ is λ -pure and $B/A \subseteq C/A$ is λ -pure, then $B \subseteq C$ is λ -pure.*
- (4) *λ -pure monomorphisms are closed under λ -directed colimits.*

Proof. (1) and (2) follow easy from the definition of λ -pure via the commutative diagram. For (3), let L be λ -presented. All we need to check is that the map $\text{Hom}_{\mathcal{G}}(L, C) \rightarrow \text{Hom}_{\mathcal{G}}(L, C/B)$ is an epimorphism. But this is just the composite

$$\text{Hom}_{\mathcal{G}}(L, C) \rightarrow \text{Hom}_{\mathcal{G}}(L, C/A) \rightarrow \text{Hom}_{\mathcal{G}}(L, (C/A)/(B/A)) \cong \text{Hom}_{\mathcal{G}}(L, C/B),$$

and these are epimorphisms by hypothesis. Finally, a proof of (4) appears in [AR94, Proposition 2.30 (1)]. \square

APPENDIX B. EXACT CATEGORIES VS. PROPER CLASSES

We show here that if \mathcal{A} is an abelian category, an exact category $(\mathcal{A}, \mathcal{E})$ in the sense of [Qui73] and [Büh10] is the same thing as a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4] and [Hov02]. See also the Historical Notes and Appendix B of [Büh10] for the equivalence to Heller's axioms for an “abelian class of short exact sequences”.

Proposition B.1. *Let \mathcal{A} be an abelian category. Then $(\mathcal{A}, \mathcal{E})$ is an exact category in the sense of [Qui73] if and only if \mathcal{E} is a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4].*

Proof. Say $(\mathcal{A}, \mathcal{E})$ is an exact category. We wish to see that \mathcal{E} is a proper class. The only thing that is not immediate from first definitions or properties of exact categories is Mac Lane's axiom (P-4), and the dual (P-4'). But abelian categories are weakly idempotent complete and so these follow from [Büh10, Proposition 7.6] which states: whenever gf is an admissible monomorphism (resp. epimorphism) then f (resp. g) is an admissible monomorphism (resp. epimorphism).

On the other hand, say \mathcal{E} is a proper class in \mathcal{A} . To see $(\mathcal{A}, \mathcal{E})$ is an exact category we just need to check the pullback/pushout axioms. But any, say pullback, exists, and pulling back along an \mathcal{E} -epimorphism p yields a diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i'} & P & \xrightarrow{p'} & C' \longrightarrow 0 \\ & & \parallel & & f' \downarrow & & f \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 0 \end{array}$$

Since i is an \mathcal{E} -monomorphism, so is $i = f'i'$. We wish to “cancel” f' to conclude i' is an \mathcal{E} -monomorphism. However, axiom (P-4) of [Mac63, Chapter XII.4] only allows this when f' is monic. But we now remedy this by imitating the argument that can be found within the proof of [Mac63, XII.4 Theorem 4.3]. First, recall that the pullback (P, f', p') can be constructed (see [Mac63, XII.4 Theorem 1.1])

so that P is the kernel in the left exact sequence $0 \rightarrow P \xrightarrow{v} B \oplus C' \xrightarrow{p\pi_1 - f\pi_2} C$ and the maps f' and p' satisfy $f' = \pi_1 v$ and $p' = \pi_2 v$. We see that

$$vi' = 1vi' = (i_1\pi_1 + i_2\pi_2)vi' = i_1(\pi_1 v)i' + i_2(\pi_2 v)i' = i_1f'i' + i_2p'i' = i_1i.$$

Since i_1 is an \mathcal{E} -monomorphism by (P-2), we see that i_1i is an \mathcal{E} -monomorphism by (P-3). So $vi' = i_1i$ is an \mathcal{E} -monomorphism, and by (P-4) we may now conclude i' is an \mathcal{E} -monomorphism. \square

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